

Weak Multiplier Hopf Algebras III

Integrals and duality

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Abstract

Let (A, Δ) be a *weak multiplier Hopf algebra* as introduced in [28] (see also [27]). It is a pair of a non-degenerate algebra A , with or without identity, and a coproduct Δ on A , satisfying certain properties. The main difference with multiplier Hopf algebras is that now, the canonical maps T_1 and T_2 on $A \otimes A$, defined by

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(c \otimes a) = (c \otimes 1)\Delta(a),$$

are *no longer* assumed to be *bijective*. Also recall that a weak multiplier Hopf algebra is called *regular* if its antipode is a bijective map from A to itself.

In this paper, we introduce and study the notion of *integrals* on such regular weak multiplier Hopf algebras. A left integral is a non-zero linear functional *on* A that is left invariant (in an appropriate sense). Similarly for a right integral.

For a regular weak multiplier Hopf algebra (A, Δ) with (sufficiently many) integrals, we construct the dual $(\hat{A}, \hat{\Delta})$. It is again a regular weak multiplier Hopf algebra with (sufficiently many) integrals. This duality extends the known duality of finite-dimensional weak Hopf algebras to this more general case. It also extends the duality of multiplier Hopf algebras with integrals, the so-called algebraic quantum groups. For this reason, we will sometimes call a regular weak multiplier Hopf algebra with enough integrals an *algebraic quantum groupoid*.

We illustrate this duality with a particular example in a separate paper (see [26]). In this paper, we only mention the main definitions and results for this example. However, we do consider the two natural weak multiplier Hopf algebras associated with a groupoid in detail and show that they are dual to each other in the sense of the above duality.

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0. Introduction

This paper continues the study of weak multiplier Hopf algebras as found in [28] and [29].

Let (A, Δ) be a *regular weak multiplier Hopf algebra* as in Definition 4.1 of [28]. The underlying algebra A is a non-degenerate algebra over \mathbb{C} and the coproduct is a homomorphism from A to $M(A \otimes A)$. It is *not* assumed to be non-degenerate, but there are certain constraints on the ranges and the kernels of the canonical maps T_1 and T_2 defined on $A \otimes A$ as

$$T_1(a \otimes b) = \Delta(a)(1 \otimes b) \quad \text{and} \quad T_2(c \otimes a) = (c \otimes 1)\Delta(a).$$

By assumption, there is a *counit* ε . This is a linear map from A to \mathbb{C} . In general it is not a homomorphism. The coproduct is assumed to be full, i.e. the legs are all of A . Consequently the counit is unique. There is a *unique antipode* S . It is an anti-algebra map and an anti-coalgebra map. Because we assume that (A, Δ) is regular, it is a bijective map from A to itself.

We refer to earlier work, in particular [28], for more details about (regular) weak multiplier Hopf algebras.

The *source* and *target maps* $\varepsilon_s : A \rightarrow M(A)$ and $\varepsilon_t : A \rightarrow M(A)$ are defined by

$$\varepsilon_s(a) = \sum_{(a)} S(a_{(1)})a_{(2)} \quad \text{and} \quad \varepsilon_t(a) = \sum_{(a)} a_{(1)}S(a_{(2)}).$$

We are using the Sweedler notation here (see further in this introduction under the item ‘Conventions and notations’). It has been carefully argued in [29] that these maps are well-defined with values in $M(A)$.

The images $\varepsilon_s(A)$ and $\varepsilon_t(A)$ are subalgebras of $M(A)$. They are called the *source* and *target algebras*. They are commuting non-degenerate subalgebras of $M(A)$. In the regular case, they embed in $M(A)$ in such a way that their multiplier algebras $M(\varepsilon_s(A))$ and $M(\varepsilon_t(A))$ still embed in $M(A)$. These multiplier algebras are denoted by A_s and A_t resp. They are still commuting subalgebras of $M(A)$.

Remark that in earlier papers on the subject, we called A_s and A_t the source and target algebras. However, in the second version of our second paper on the subject [29], we changed the terminology. For more details, see [29].

The examples associated with a groupoid

Consider a groupoid G . One can associate two regular weak multiplier Hopf algebras.

First there is the algebra A , defined as the space $K(G)$ of complex functions on G with finite support and pointwise product. The coproduct Δ on $K(G)$ is defined by

$$\Delta(f)(p, q) = \begin{cases} f(pq) & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

In this example, the algebra A_s is the algebra of functions on G so that $f(p) = f(q)$ whenever $p, q \in G$ satisfy $s(p) = s(q)$ (where s is the source map of the groupoid). The

algebra A_t consists of functions f on G so that $f(p) = f(q)$ if $t(p) = t(q)$ for $p, q \in G$ (where t is the target map).

For the *second case*, we take the algebra B , defined as the groupoid algebra $\mathbb{C}G$ of G . If we use $p \mapsto \lambda_p$ for the canonical embedding of G in $\mathbb{C}G$, then for $p, q \in G$ we have $\lambda_p \lambda_q = \lambda_{pq}$ if pq is defined and $\lambda_p \lambda_q = 0$ otherwise. The coproduct is defined by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all $p \in G$. Here the multiplier algebras B_s and B_t of source and target algebras coincide and it is the multiplier algebra of the span of elements of the form λ_e where e is a unit of G .

Remark that the first algebra has no identity if G is infinite. The second one has no identity if the set of units of G is infinite. In particular, only when there is an identity, they are weak Hopf algebras. If not, they are weak *multiplier* Hopf algebras. We refer to Examples 1.15 and 1.16 in [28] and Example 3.1 in [29] for more details on these examples.

These two cases are *dual* to each other. The duality is given by $\langle f, \lambda_p \rangle = f(p)$ whenever $f \in K(G)$ and $p \in G$. We will give more details about this duality in Section 3 of this paper where we illustrate the duality for regular weak multiplier Hopf algebras with integrals as developed in Section 2.

Let us now look at *integrals* for these two examples.

First take for A the algebra $K(G)$. Let g be any complex function on G satisfying $g(q) = g(q')$ whenever q and q' have the same source (i.e. we have $g \in A_s$). Define a linear functional φ on A by $\varphi(f) = \sum_q g(q)f(q)$. Then φ will be a left integral and all left integrals are of this form. We refer to Section 1 for the precise definition of a left integral and to Section 3 for the proof of this result. Similarly, a right integral ψ is a linear functional on A of the form $\psi(f) = \sum_q h(q)f(q)$ where now h is a function such that $h(q) = h(q')$ whenever q and q' have the same target, i.e. when $h \in A_t$. Again we refer to Section 3 for a proof of this result. It can happen that φ will be left and right invariant (e.g. if $g = h = 1$) but not all left integrals will also be right integrals.

Next consider the groupoid algebra $\mathbb{C}G$. Let g be a complex function on G with support in the set of units. Define $\varphi(\lambda_p) = g(p)$ for all p . This will give a left integral and all left integrals are of this form. Now the left and right integrals are the same because the coproduct is coabelian.

Again, we refer to Section 3 where we will treat these examples in detail.

What we study in this paper generalizes the results on integrals, described above for the regular weak multiplier Hopf algebras $K(G)$ and $\mathbb{C}G$, associated with a groupoid G , to general regular weak multiplier Hopf algebras.

Content of the paper.

In *Section 1*, we introduce the notion of left and right integrals on regular weak multiplier Hopf algebras. We discuss the definition and compare with the literature on weak Hopf algebras and other cases where integrals are studied.

To do this, we first prove some basic properties and use them to give alternative definitions of left and right invariance of linear functionals. Then we prove properties that are very similar as in the case of multiplier Hopf algebras with integrals. However, we state clearly

what the main differences are. We do not have automatically that a non-zero integral is faithful and so in order to be able to get nice results, we need to assume that there are *enough* integrals. If there is a single faithful integral, we get the scaling constant, the modular element as well as the modular automorphisms, just as for multiplier Hopf algebras. In the general case, one can probably get partial results along these lines. We refer to Section 4 where we discuss this further.

In this section we already define the dual space \hat{A} and give some characterizations of it. However, it is only in Section 2 that we make this dual space again into a weak multiplier Hopf algebra. The basic examples are considered in Section 3.

In *Section 2* we study duality. We show that the dual \hat{A} of A , already introduced as a subspace of the dual space A' in Section 1, can be equipped with a product and a coproduct, adjoint to the coproduct and product on A respectively. This turns \hat{A} again into a regular weak multiplier Hopf algebra.

The construction of the coproduct on the dual is not so obvious. We show that the adjoints of the various canonical maps and their generalized inverses of A can be found on the dual \hat{A} . This provides the coproduct on \hat{A} . We show that indeed, this is adjoint to the product on A by first extending the pairings to the multiplier algebras (on one side).

We obtain explicit formulas for the dual objects: the counit, the canonical idempotent, the antipode and the source and target maps and source and target algebras. In particular, we find that the source algebra of \hat{A} is isomorphic with the target algebra of A and that the target algebra of \hat{A} is isomorphic with the source algebra of A . This is known for weak Hopf algebras, see e.g. Lemma 2.6 in [2].

Finally, we construct integrals on the dual and show that there is a faithful set of integrals, also on the dual. Then in turn, the dual of $(\hat{A}, \hat{\Delta})$ can be constructed. It is canonically isomorphic with the original weak multiplier Hopf algebra (A, Δ) .

In *Section 3* we discuss examples.

First we look at the two weak multiplier Hopf algebras associated with a groupoid G . The first one is the function algebra $K(G)$ while the second one is the groupoid algebra $\mathbb{C}G$. The two weak multiplier Hopf algebras have been discussed in earlier papers, see e.g. Examples 1.15 and 1.16 in [28]. Here we obtain the integrals. It turns out that in the two cases, there is a faithful left integral that is also right invariant. We also show in detail how the groupoid algebra is the dual of the function algebra, hereby applying the general duality theory as developed in Section 2.

We also briefly treat the weak multiplier Hopf algebra associated with a separability idempotent. This example is already considered in Proposition 2.8 of [29]. Here, we find the integrals on this weak multiplier Hopf algebra and again, we follow the results of Section 2 to obtain the dual of this weak multiplier Hopf algebra. We only formulate the main results about this case. We refer to a separate paper ([26]) where we treat this example in great detail. By doing so, we illustrate various aspects of the theory as developed in this paper.

In *Section 4* we draw some conclusions and discuss possible further research.

Conventions and notations

We only work with algebras A over \mathbb{C} (although we believe that this is not essential). We do not assume that they are unital but we need that the product is non-degenerate. Our algebras are all idempotent (that is $A^2 = A$). In some situations, this is a condition, while at other places, it follows from the other axioms. In certain parts of the paper, we even need the algebras to have local units. Also this is sometimes a condition, while in other cases, it is a consequence. Then of course, the product is automatically non-degenerate and also the algebra is idempotent.

When A is such an algebra, we use $M(A)$ for the multiplier algebra of A . When m is in $M(A)$, then by definition we have elements am and mb in A for all $a, b \in A$ and $(am)b = a(mb)$. The algebra A sits in $M(A)$ as an essential two-sided ideal and $M(A)$ is the largest algebra with identity having this property.

We consider $A \otimes A$, the tensor product of A with itself. It is again an idempotent, non-degenerate algebra and we can consider the multiplier algebra $M(A \otimes A)$. The same is true for a multiple tensor product. We will no longer use σ for the flip map on $A \otimes A$ (as we have done sometimes in earlier papers) because this will be used now for the modular automorphism. In stead, we will use ζ for the flip map on $A \otimes A$, as well as for its natural extension to $M(A \otimes A)$.

We use 1 for the identity in any of these multiplier algebras. On the other hand, we mostly use ι for the identity map on A (or other spaces), although sometimes, we also write 1 for this map. The identity element in a group is denoted by e . If G is a groupoid, we will also use e for units. Units are considered as being elements of the groupoid and we use s and t for the source and target maps from G to the set of units.

The space of all linear functionals on A is denoted by A' . A linear functional ω on an algebra A is called *faithful* if given $a \in A$ we have $a = 0$ if either $\omega(ab) = 0$ for all $b \in A$ or $\omega(ba) = 0$ for all $b \in A$.

When A is an algebra, we denote by A^{op} the algebra obtained from A by reversing the product. When Δ is a coproduct on A , we denote by Δ^{cop} the coproduct on A obtained by composing Δ with the flip map ζ .

For a coproduct Δ , as we define it in Definition 1.1 of [28], we assume that $\Delta(a)(1 \otimes b)$ and $(c \otimes 1)\Delta(a)$ are in $A \otimes A$ for all $a, b, c \in A$. This allows us to make use of the *Sweedler notation* for the coproduct. The reader who wants to have a deeper understanding of this, is referred to [25] where the use of the Sweedler notation for coproducts that do not map into the tensor product, but rather in its multiplier algebra, is explained in detail.

Basic references

For the theory of Hopf algebras, we refer to the standard works of Abe [1] and Sweedler [16]. For multiplier Hopf algebras and integrals on multiplier Hopf algebras, we refer to [22] and [23]. Weak Hopf algebras have been studied in [2] and [3] and more results are found in [11] and [12]. Various other references on the subject can be found in [21]. In particular, we refer to [13] because we will use notations and conventions from this paper when dealing with weak multiplier Hopf algebras.

Weak multiplier Hopf algebras have been introduced in [28]. See also [27] for a preliminary

paper on the subject. The source and target algebras and the source and target maps for a weak multiplier Hopf algebra are studied in detail in [29]. In the more recent second version of the paper, more results are obtained for possibly non-regular weak multiplier Hopf algebras. Integrals on regular weak multiplier Hopf algebras are also considered in [8] where a form of the Larson-Sweedler theorem is obtained in the framework of regular weak multiplier Hopf algebras.

Finally, for the theory of groupoids, we refer to [4], [6], [14] and [15].

Note added while preparing this manuscript

While preparing this manuscript, we came across recent work by Thomas Timmermann. He studies integrals and duality for regular multiplier Hopf algebroids in [18] and [19]. The relation and possible overlap between his work and this one has to be investigated further. See also a remark in Section 4.

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1. Integrals on weak multiplier Hopf algebras

In this section, we treat integrals on regular weak multiplier Hopf algebras. We will make a remark about the theory of integrals on possibly non-regular weak multiplier Hopf algebras in Section 4.

Integrals on weak multiplier bialgebras have been studied before in the context of a version of the Larson-Sweedler theorem for weak multiplier Hopf algebras, see [8]. In the context of multiplier Hopf algebroids, integrals were defined and investigated in [18]. Here, we give a self-contained treatment.

We fix a *regular* weak multiplier Hopf algebra (A, Δ) . We start with the definition of a left, respectively a right integral on A . We first prove some basic properties that give equivalent definitions for left and right invariance of linear functionals. We compare our definition with the ones found in the literature for weak Hopf algebras (and other cases).

Then we develop the theory of integrals further. We obtain results that are very similar as those for multiplier Hopf algebras. The main difference is that integrals are not necessarily unique and they are not automatically faithful. Therefore, to obtain a useful theory, we will need to assume not only the existence of integrals, but we have to require that there are enough integrals, in a sense to be explained.

Left and right integrals

Before we give the main definitions, recall that for any element $a \in A$ and any linear functional ω on A , we can define a multiplier $x \in M(A)$ by

$$xb = (\iota \otimes \omega)(\Delta(a)(b \otimes 1)) \quad \text{and} \quad bx = (\iota \otimes \omega)(b \otimes 1)\Delta(a)$$

where b is in A . This multiplier x is denoted as $(\iota \otimes \omega)\Delta(a)$. Similarly we can define $(\omega \otimes \iota)\Delta(a)$ in $M(A)$ for $a \in A$ and any linear functional ω on A . We need regularity of the coproduct to do this but this is satisfied as we work with a regular weak multiplier Hopf algebra.

The above property is used in the following definition.

1.1. Definition A linear functional $\varphi : A \rightarrow \mathbb{C}$ is called *left invariant* if $(\iota \otimes \varphi)\Delta(a) \in A_t$ for all $a \in A$. Similarly, a linear functional ψ on A is called *right invariant* if $(\psi \otimes \iota)\Delta(a) \in A_s$ for all $a \in A$. A non-zero left invariant functional is called a *left integral* and a non-zero right invariant functional is called a *right integral* on the weak multiplier Hopf algebra (A, Δ) .

Recall that A_t and A_s are subspaces of $M(A)$ and so the above definition makes sense.

1.2. Remark i) If (A, Δ) is actually a multiplier Hopf algebra, so that A_t and A_s are nothing else but scalar multiples of the identity, it follows from the requirement in the definition, that for any $a \in A$ we have $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ when φ is left invariant and $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ when ψ is right invariant. So, the above definition is consistent with the definition of integrals in the case of a regular multiplier Hopf algebra.

ii) As the antipode S flips the coproduct and maps A_t to A_s and vice versa, we will have that $\varphi \circ S$ is a right integral when φ is a left integral and that $\psi \circ S$ is a left integral when ψ is a right integral. Also $\varphi \circ S^2$ will be a left integral when φ is a left integral. However, as we no longer have uniqueness of integrals, we can not expect that this is a scalar multiple of φ . We will see later how this property is modified as a consequence of Proposition 1.8, in the case of a faithful integral.

If (A, Δ) is a weak Hopf algebra, then we have $\varepsilon_t(x) = x$ for all $x \in A_t$ and so for all a we have $(\iota \otimes \varphi)\Delta(a) = (\varepsilon_t \otimes \varphi)\Delta(a)$ when φ is left invariant. This shows that our definition of a left integral coincides with that of a left integral on a weak Hopf algebra as it is found in [2]. Similarly for a right integral.

However, we cannot simply import the definition of integrals as defined for weak Hopf algebras in this setting. It would not make sense to write $(\iota \otimes \varphi)\Delta(a) = (\varepsilon_t \otimes \varphi)\Delta(a)$ because we have not defined ε_t on the multiplier algebra. Nevertheless, this just turns out to be a matter of interpretation as we will see in the next proposition.

1.3. Proposition Let φ be a left integral on A . Then

$$(\iota \otimes \varphi)\Delta(a) = \sum_{(a)} a_{(1)} S(a_{(2)}) \varphi(a_{(3)}) \quad (1.1)$$

for all $a \in A$. Similarly, when ψ is a right integral, we have

$$(\psi \otimes \iota)\Delta(a) = \sum_{(a)} \psi(a_{(1)}) S(a_{(2)}) a_{(3)} \quad (1.2)$$

for all a .

Proof: The formula (1.1) is given a meaning if we multiply with an element c of A from the left. The left hand side becomes an element in A and this is true also for the right hand side. Indeed, the element c will first cover $a_{(1)}$ and subsequently, also $a_{(2)}$ will be covered. See e.g. Remark 1.2 in [29] about coverings in these formulas. The formula (1.2) is given a meaning if we multiply with an element of A from the right.

To prove the formula (1.1) take $a, b, c \in A$ and let $x = (\iota \otimes \varphi)\Delta(a)$. By definition, we know that $x \in A_t$ and therefore, by a property of the target map (see Proposition 2.7 in [29]), we have that $x\varepsilon_t(b) = \varepsilon_t(xb)$. We multiply this equality with c from the left and use the definition of ε_t . This gives

$$\begin{aligned} cx\varepsilon_t(b) &= c\varepsilon_t(xb) = \sum_{(a),(b)} ca_{(1)}b_{(1)}S(a_{(2)}b_{(2)})\varphi(a_{(3)}) \\ &= \sum_{(a),(b)} ca_{(1)}b_{(1)}S(b_{(2)})S(a_{(2)})\varphi(a_{(3)}) \\ &= \sum_{(a)} ca_{(1)}\varepsilon_t(b)S(a_{(2)})\varphi(a_{(3)}). \end{aligned}$$

Now think of $ca_{(1)} \otimes a_{(2)}$ as $p \otimes q$ in $A \otimes A$. For the right hand side of the last expression above we then consider

$$p\varepsilon_t(b)S((\iota \otimes \varphi)\Delta(q)).$$

As φ is left invariant, we know that $(\iota \otimes \varphi)\Delta(q)$ is in A_t and so $S((\iota \otimes \varphi)\Delta(q))$ will be in A_s . Now we use that A_t and A_s commute. This implies that

$$p\varepsilon_t(b)S((\iota \otimes \varphi)\Delta(q)) = pS((\iota \otimes \varphi)\Delta(q))\varepsilon_t(b).$$

Replace again $p \otimes q$ by $ca_{(1)} \otimes a_{(2)}$ and insert the result in the original formula above. This will give

$$cx\varepsilon_t(b) = \sum_{(a)} ca_{(1)}S(a_{(2)})\varepsilon_t(b)\varphi(a_{(3)}).$$

As this is true for all b , we find $cx = \sum_{(a)} ca_{(1)}S(a_{(2)})\varphi(a_{(3)})$ for all c (as a consequence of Proposition 2.10 in [29]). This gives the correct interpretation of the formula (1.1) as we want to prove it.

The argument for the right integral is completely similar. \square

The formulas (1.1) and (1.2) in the previous proposition suggest that $(\iota \otimes \varphi)\Delta(a) \in \varepsilon_t(A)$ when φ is a left integral and that $(\psi \otimes \iota)\Delta(a) \in \varepsilon_s(A)$ when ψ is a right integral. We will see shortly that this is indeed the case. However, we can not conclude it from the above formulas.

These results are intimately related with the following result obtained in [8].

1.4. Proposition Denote

$$\begin{aligned} F_1 &= (\iota \otimes S)E & \text{and} & & F_3 &= (\iota \otimes S^{-1})E \\ F_2 &= (S \otimes \iota)E & \text{and} & & F_4 &= (S^{-1} \otimes \iota)E. \end{aligned}$$

Then, if φ is a left integral and if ψ is a right integral, we have for all a in A ,

$$(\iota \otimes \varphi)\Delta(a) = (\iota \otimes \varphi)(F_2(1 \otimes a)) = (\iota \otimes \varphi)((1 \otimes a)F_4) \quad (1.3)$$

$$(\psi \otimes \iota)\Delta(a) = (\psi \otimes \iota)((a \otimes 1)F_1) = (\psi \otimes \iota)(F_3(a \otimes 1)). \quad (1.4)$$

The results are found in Proposition 2.7 of [8] and the proofs can be found in that paper. The context of [8] is a little different, but this has no effect for the proofs.

We will now continue the study of left and right integrals on a regular weak multiplier Hopf algebra. We aim at similar properties as for regular multiplier Hopf algebras. The following result is, surprisingly enough, precisely the same as in that case.

1.5. Proposition Let φ be a linear functional on A . Given $a, b \in A$, define

$$c = (\iota \otimes \varphi)(\Delta(a)(1 \otimes b)) \quad \text{and} \quad d = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)). \quad (1.5)$$

These elements belong to A . Then φ is left invariant if and only if we have $d = S(c)$ for all a and b . Similarly, let ψ be a linear functional on A and now define, given $a, b \in A$ the elements $c, d \in A$ by

$$c = (\psi \otimes \iota)((a \otimes 1)\Delta(b)) \quad \text{and} \quad d = (\psi \otimes \iota)(\Delta(a)(b \otimes 1)). \quad (1.6)$$

Then ψ is right invariant if and only if $d = S(c)$ for all a and b .

Proof: i) First assume that φ is left invariant. Take any a and b in A and define the elements c and d as in the formulation of the proposition. We have

$$(1 \otimes a)\Delta(b) = \sum_{(a)} (S(a_{(1)}) \otimes 1)\Delta(a_{(2)}b) \quad (1.7)$$

and if we apply φ on the second factor, we find

$$d = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)) = \sum_{(a)} S(a_{(1)})((\iota \otimes \varphi)\Delta(a_{(2)}b)).$$

Now apply Δ . Because $\Delta(px) = \Delta(p)(x \otimes 1)$ when $p \in A$ and $x \in A_t$, it follows from the left invariance of φ that

$$\Delta(d) = \sum_{(a)} \Delta(S(a_{(1)}))((\iota \otimes \varphi)\Delta(a_{(2)}b) \otimes 1).$$

Next multiply with an element q from the right in the second factor. If we use that S flips the coproduct, we find

$$\begin{aligned} \Delta(d)(1 \otimes q) &= \sum_{(a)} (S(a_{(2)}) \otimes S(a_{(1)})q)((\iota \otimes \varphi)\Delta(a_{(3)}b) \otimes 1) \\ &= \sum_{(a)} S(a_{(2)})((\iota \otimes \varphi)\Delta(a_{(3)}b)) \otimes S(a_{(1)})q. \end{aligned}$$

We use the formula (1.7) with a replaced by $a_{(2)}$ and we find

$$\Delta(d)(1 \otimes q) = \sum_{(a)} (\iota \otimes \varphi)((1 \otimes a_{(2)})\Delta(b)) \otimes S(a_{(1)})q.$$

We can now safely apply ε on the first leg of this equality and we get precisely $dq = S(c)q$ where $c = \sum_{(a)} a_{(1)}\varphi(a_{(2)}b)$. We can cancel q and this proves one part of the proposition.

ii) Conversely, assume that φ is a linear functional satisfying the requirement in the proposition. Then for all p, q in A we have

$$\begin{aligned} (\iota \otimes \varphi)\Delta(pq) &= \sum_{(p)} p_{(1)}(\iota \otimes \varphi)((1 \otimes p_{(2)})\Delta(q)) \\ &= \sum_{(p)} p_{(1)}S((\iota \otimes \varphi)(\Delta(p_{(2)})(1 \otimes q))) \\ &= \sum_{(p)} p_{(1)}S(p_{(2)})\varphi(p_{(3)}q). \end{aligned}$$

We can cover these expressions if we multiply with an element of A from the left. As $A^2 = A$, this proves that $(\iota \otimes \varphi)\Delta(a) \in \varepsilon_t(A)$ for all $a \in A$. And because $\varepsilon_t(A)$ is a subset of A_t , we have shown that φ is left invariant.

iii) The proof for right invariant functionals is completely similar. \square

We have made use of the Sweedler notation but most of the time, we have used proper coverings. At other places where covering is necessary, we have indicated how the expressions can be covered. The reader can verify that this can easily be done.

In the case of a weak Hopf algebra, that is when A has an identity, we can take $a = 1$ in Equation (1.5) and we will obtain the first equality of Equation (1.3). Similarly, if we take $b = 1$ in Equation (1.6) we will get the first equation in (1.4). Doing this with $b = 1$ in the first case and with $a = 1$ in the second case, we get the other equalities.

At the end of the proof of the previous proposition, we have found that

$$(\iota \otimes \varphi)(\Delta(pq)) = \sum_{(p)} p_{(1)}S(p_{(2)})\varphi(p_{(3)}q) = \sum_{(p)} \varepsilon_t(p_{(1)})\varphi(p_{(2)}q)$$

for all $p \in A$ when φ is a left integral. If the algebra has an identity, as in the case of weak Hopf algebras, we can take $q = 1$ and this will give again the formula obtained in Proposition 1.3. In general, it follows that $(\iota \otimes \varphi)(\Delta(a)) \in \varepsilon_t(A)$ when $a \in A$. This result was expected but not yet proven (see a remark following Proposition 1.3). Of course, when ψ is a right integral, we will have $(\psi \otimes \iota)\Delta(a) \in \varepsilon_s(A)$.

Remark in passing that in the theory of *algebraic quantum hypergroups* (see e.g. [5]), the above formulas are still true for left and right integrals. In fact, they are used to define integrals in that theory. In the theory of quantum hypergroups, it is not assumed that the coproduct is an algebra homomorphism.

Having exactly the same characterizing formulas for invariant functionals (Proposition 1.5), one can expect that many of the other properties of integrals, as we know them in the theory of multiplier Hopf algebras and algebraic quantum hypergroups, will also be true here. We will see that this is the case, except for the uniqueness and the related property of faithfulness.

The claim above is nicely illustrated in the following proposition where we prove a set of formulas that will be crucial for the various properties of the integrals, as well as for the definition of the dual \widehat{A} and to show that it is again a regular weak multiplier Hopf algebra (cf. Section 2).

1.6. Proposition Let φ be any left integral and ψ any right integral on A . In the following formulas, p and q are arbitrary elements in A .

i) We have $\psi(xa) = \varphi(xb)$ for all $x \in A$ if

$$\begin{aligned} a &= (\iota \otimes \varphi)((\iota \otimes S)(\Delta(p))(1 \otimes q)) \\ b &= (\psi \otimes \iota)((S^{-1} \otimes \iota)(\Delta(q))(p \otimes 1)). \end{aligned}$$

ii) We have $\psi(ax) = \varphi(bx)$ for all $x \in A$ if

$$\begin{aligned} a &= (\iota \otimes \varphi)((1 \otimes q)(\iota \otimes S^{-1})\Delta(p)) \\ b &= (\psi \otimes \iota)((p \otimes 1)(S \otimes \iota)\Delta(q)). \end{aligned}$$

iii) We have $\psi(xa) = \varphi(bx)$ for all $x \in A$ if

$$\begin{aligned} a &= (\iota \otimes \varphi)((1 \otimes q)(\iota \otimes S)\Delta(p)) \\ b &= (\psi \otimes \iota)((S \otimes \iota)(\Delta(q))(p \otimes 1)). \end{aligned}$$

iv) We have $\psi(ax) = \varphi(xb)$ for all $x \in A$ if

$$\begin{aligned} a &= (\iota \otimes \varphi)((\iota \otimes S^{-1})(\Delta(p))(1 \otimes q)) \\ b &= (\psi \otimes \iota)((p \otimes 1)(S^{-1} \otimes \iota)\Delta(q)). \end{aligned}$$

Proof: To prove i), take $p, q \in A$ and also $x \in A$. Consider the expression $(\psi \otimes \varphi)((\Delta(x)(p \otimes q))$. Using the formula obtained in Proposition 1.5 for φ with a, b replaced by p, q , we find that

$$\begin{aligned} (\psi \otimes \varphi)(\Delta(x)(p \otimes q)) &= \psi(\cdot p)((\iota \otimes \varphi)(\Delta(x)(1 \otimes q))) \\ &= \psi(\cdot p)(S^{-1}((\iota \otimes \varphi)((1 \otimes x)\Delta(q)))) \\ &= (\psi \otimes \varphi)((1 \otimes x)(S^{-1} \otimes \iota)(\Delta(q))(p \otimes 1)) \\ &= \varphi(xb) \end{aligned}$$

where $b = (\psi \otimes \iota)((S^{-1} \otimes \iota)(\Delta(q))(p \otimes 1))$. On the other hand, if we use the formula in Proposition 1.5 for ψ , we find that

$$\begin{aligned} (\psi \otimes \varphi)(\Delta(x)(p \otimes q)) &= \varphi(\cdot q)((\psi \otimes \iota)(\Delta(x))(p \otimes 1)) \\ &= \varphi(\cdot q)(S((\psi \otimes \iota)(x \otimes 1)\Delta(p))) \\ &= (\psi \otimes \varphi)((x \otimes 1)(\iota \otimes S)(\Delta(p))(1 \otimes q)) \\ &= \psi(xa) \end{aligned}$$

where $a = (\iota \otimes \varphi)((\iota \otimes S)(\Delta(p))(1 \otimes q))$. This proves i).

To prove ii) we start with the expression $(\psi \otimes \varphi)((p \otimes q)\Delta(x))$ and proceed as above. For the proof of iii), we take $(\psi \otimes \varphi)((1 \otimes q)\Delta(x)(p \otimes 1))$ and for iv) we finally start with $(\psi \otimes \varphi)((p \otimes 1)\Delta(x)(1 \otimes q))$. And also in these cases, we use the two formulas of Proposition 1.5. \square

The case of a single faithful integral

In the event that there exists a single faithful integral, we find some nice consequences of these results.

First observe the following. Assume that φ is just a faithful linear functional. We claim that all elements in A are linear combinations of elements of the form a in each of the items in the previous proposition.

Indeed, suppose e.g. that ω is a linear functional on A that vanishes on all elements of the form

$$(\iota \otimes \varphi)((\iota \otimes S)(\Delta(p))(1 \otimes q))$$

as considered in item i) of the proposition. Replace q by qq' and write

$$x = (\omega \otimes \iota)((\iota \otimes S)(\Delta(p))(1 \otimes q)).$$

Then $\varphi(xq') = 0$ for all q' and because φ is assumed to be faithful, it follows that $x = 0$. This is true for all p, q and because the coproduct is full, we must have $\omega = 0$. This proves the claim. Recall that fullness of the coproduct means that the legs are all of A .

A similar argument can be used for the three other cases.

As a first consequence, we get the KMS property.

1.7. Proposition If φ is a faithful left integral, there exists an automorphism σ of A satisfying $\varphi(ab) = \varphi(b\sigma(a))$ for all a, b . It leaves φ invariant.

Proof: Because there is a faithful left integral φ , there is also a faithful right integral ψ . If we combine the formulas in the items i) and iii) of Proposition 1.6, for these faithful integrals, we find that for any element $a \in A$ there is an element $b \in A$ so that $\varphi(ax) = \varphi(xb)$ for all x . The element b is uniquely determined and we write it as $\sigma(a)$. We clearly have a linear map satisfying

$$\varphi(x\sigma(aa')) = \varphi(aa'x) = \varphi(a'x\sigma(a)) = \varphi(x\sigma(a)\sigma(a'))$$

so that $\sigma(aa') = \sigma(a)\sigma(a')$ by the faithfulness of φ . The faithfulness also implies that σ is injective. And a similar argument to find σ will also give that it is surjective. Hence, σ is an automorphism of A .

Finally, if $a, b \in A$ we find

$$\varphi(ab) = \varphi(b\sigma(a)) = \varphi(\sigma(a)\sigma(b)) = \varphi(\sigma(ab)).$$

Because A is idempotent, it follows that φ is invariant under σ . □

We call it the *modular automorphism* for φ .

Observe that any faithful linear functional on a finite-dimensional algebra admits a modular automorphism. In the infinite-dimensional case however, not all faithful linear functionals will admit a modular automorphism. So, in order to obtain the result above, we really need more than just a faithful linear functional.

In the literature, it is called the Nakayama automorphism (or rather its inverse), see e.g. [10]. The terms KMS property and modular automorphisms find their origin in the theory of operator algebras and their usage in mathematical physics, see e.g. Chapter VIII in [17].

A next result gives the relation between left integrals.

1.8. Proposition Assume that φ and φ_1 are left integrals and that φ is faithful. Then there is an element y in A_s so that $\varphi_1(x) = \varphi(xy)$ for all x .

Proof: If we combine item i) of Proposition 1.6 with a single faithful right integral and these two left integrals, we find that for all a , there is an element b so that $\varphi_1(xa) = \varphi(xb)$ for all x . We must have that $b = ay$ for a right multiplier y of A . Then $\varphi_1(x) = \varphi(xy)$ for all x . In a similar way, we find a left multiplier y' so that $\varphi_1(x) = \varphi(y'x)$ for all x . In particular $\varphi(xy) = \varphi(y'x)$ for all x .

If we replace x by ab and we get

$$\varphi(ay\sigma(b)) = \varphi(bay) = \varphi(y'ba) = \varphi(a\sigma(y'b)).$$

Replace a by $a'a$ and use that this is true for all a' . It follows that $ay\sigma(b) = a\sigma(y'b)$ for all a, b . This implies that $y \in M(A)$ and that $y = \sigma(y')$.

Now we apply Proposition 1.5 for φ_1 and for φ . We find

$$(\iota \otimes \varphi)((1 \otimes a)\Delta(by)) = S((\iota \otimes \varphi)(\Delta(a)(1 \otimes by))) = (\iota \otimes \varphi)((1 \otimes a)\Delta(b)(1 \otimes y)).$$

By the faithfulness of φ we get $\Delta(by) = \Delta(b)(1 \otimes y)$ for all b . This implies that y is in A_s . \square

There is also a converse result. If φ is any left integral and if $y \in A_s$, then $\varphi(\cdot y)$ as well as $\varphi'(y \cdot)$, are again left invariant linear functionals.

As a consequence, this will imply that σ will leave A_s globally invariant.

Also, we find that there is a distinguished invertible element $y \in A_s$ satisfying $\varphi(S^2(x)) = \varphi(xy)$ for all x .

Using completely similar methods, we can prove the following formula for a right integral in terms of a faithful left integral.

1.9. Proposition Let φ be a faithful left integral and ψ any right integral. Then there is an element $\delta \in M(A)$ such that $\psi(x) = \varphi(x\delta)$ for all x . If ψ is also faithful, δ is invertible in $M(A)$.

In many examples (as we will see e.g. in Section 3) a single faithful integral exists. However, there are known examples where enough integrals exist (in the sense of our Definition 1.11 below), but not a single faithful one. See e.g. Proposition 2.5 in [7].

Still, it is expected that the results above, proven in the case where a single faithful integral exists, will also have similar counterparts in the general situation.

We will discuss this further in Section 4.

The dual space \hat{A}

We will now define the subspace \hat{A} of the dual space A' . Before we do that, we show in the following example that we can not expect that integrals on a weak multiplier Hopf algebra are automatically faithful (as in the case of multiplier Hopf algebras). This will be important for the upcoming definition of \hat{A} in this setting.

1.10. Example Take two Hopf algebras (B, Δ_B) and (C, Δ_C) . Let A be the direct sum of the algebras B and C . So elements in A are pairs (b, c) with $b \in B$ and $c \in C$, with pointwise operations. We consider B and C as sitting in A via the homomorphisms $b \mapsto (b, 0)$ and $c \mapsto (0, c)$. The element $(1, 0)$ is denoted by e and then $(0, 1)$ is $1 - e$.

Define a coproduct Δ on A by the requirement that it coincides with Δ_B on B and with Δ_C on C . The pair (A, Δ) is no longer a Hopf algebra, but a weak Hopf algebra. Indeed we get

$$\Delta(1) = \Delta(e) + \Delta(1 - e) = e \otimes e + (1 - e) \otimes (1 - e).$$

We see that $\Delta(1)$ is strictly smaller than $1 \otimes 1$ in $A \otimes A$.

If one of the components has integrals while the other has not, then we will have an example of a weak Hopf algebra with integrals, but obviously not enough integrals. More precisely, there will be no faithful integral.

So indeed, as an immediate consequence we see from this example that it will not be possible to show that an integral is automatically faithful as in the case of multiplier Hopf algebras. The example also makes clear that it will not be sufficient to assume the existence of integrals. We need *enough* integrals in the sense of the following definition.

1.11. Definition We say that a weak multiplier Hopf algebra has a *faithful set of integrals* if the following two conditions are satisfied. Given an element $x \in A$, we must have $x = 0$ if $\varphi(xa) = 0$ for all left integrals φ and elements $a \in A$. Similarly also if $\varphi(ax) = 0$ for all left integrals φ and elements $a \in A$, then $x = 0$.

If there is only one left integral (up to a scalar), as in the case of multiplier Hopf algebras, this condition is the same as faithfulness of this left integral. On the other hand, if there is a faithful left integral, then the condition above is fulfilled. So, what we essentially do with this definition is generalizing the requirement that there is a faithful left integral (or simply that there is an integral in the case of multiplier Hopf algebras).

Remark that we can formulate the condition also in terms of right integrals by applying the antipode.

Now, we are ready to give and discuss the definition of the *dual space* \hat{A} as follows.

1.12. Definition Assume that (A, Δ) is a regular weak multiplier Hopf algebra with a faithful set of integrals. Then we define \hat{A} as the space of linear functionals on A spanned by elements of the form $\varphi(\cdot a)$ where φ is a left integral and $a \in A$.

The choice of the representation of the elements in \hat{A} is not important as we will now prove, using the results of Proposition 1.6.

1.13. Proposition For any left integral φ and right integral ψ and for all elements $a \in A$, we have that the linear functionals of the form $\varphi(\cdot a)$, $\varphi(a \cdot)$, $\psi(\cdot a)$ en $\psi(a \cdot)$ are all in \widehat{A} . Moreover \widehat{A} is the linear span of functionals of one of these four types, where φ , respectively ψ run over all left, respectively right integrals and a over all elements in A .

Proof: By the assumption the set of left integrals is faithful as in Definition 1.11. If we apply the antipode, we also find that the set of right integrals is faithful.

We claim that all elements in A can be obtained as a linear span of elements of one of the eight forms as we find them in the formulation of Proposition 1.6. Let us show this for elements of the first form, namely

$$(\iota \otimes \varphi)((\iota \otimes S)(\Delta(p))(1 \otimes q))$$

where $p, q \in A$. If this is not be true, there exists a non-zero linear functional ω on A that is 0 on all such elements. For such a functional ω , we have that $\varphi(S(r)q) = 0$ with $r = (\omega \otimes \iota)\Delta(p)$ for all $p, q \in A$ and for any left integral φ . Replace q by a product qq' and use the faithfulness of the set of left integrals to obtain that then $S(r)q = 0$ for all p, q . Using the bijectivity of the antipode we obtain that $(\omega \otimes \iota)((1 \otimes q)\Delta(p)) = 0$, again for all p, q . From the fullness of the coproduct, it follows that $\omega = 0$ and this is a contradiction. A similar argument is possible for all the other cases as we know that also the set of right integrals is faithful.

Then, using the results of Proposition 1.6, it not only follows that elements of the four forms in the proposition all belong to \widehat{A} , but also that \widehat{A} is the span of such elements for each of these forms. \square

In what follows, we will use the following terminology.

1.14. Definition Let (A, Δ) be a regular weak multiplier Hopf algebra. If there exists a faithful set of left integrals, we call it an *algebraic quantum groupoid* (or a *weak algebraic quantum group*).

Recall that we also use the term *algebraic quantum group* for a *regular multiplier Hopf algebra with integrals*. The terminology is therefore consistent.

Remark that the term *algebraic quantum groupoid* is also used in [19] within the context of multiplier Hopf algebroids. The notion is different from the one we introduce above in Definition 1.14.

In Section 3, where we treat duality, we will use the above results to show that the adjoints of the product and coproduct of A make \widehat{A} again into a regular weak multiplier Hopf algebra with a faithful set of integrals. In other words, the dual of an algebraic quantum groupoid is again an algebraic quantum groupoid (see Theorem 2.18 in Section 2).

2. Duality

Let (A, Δ) be a regular weak multiplier Hopf algebra. Assume that there is a faithful set of integrals (as defined in Definition 1.11). So, using the terminology as introduced in Section 1, we assume that (A, Δ) is an *algebraic quantum groupoid* (cf. Definition 1.14).

Recall the definition of the dual \hat{A} . It is the space of linear functionals spanned by elements of the form $\varphi(\cdot a)$ where φ is a left integral and a an element of A (see Definition 1.12). Remember that we also can take elements of the form $\varphi(a \cdot)$ and that left integrals can be replaced by right integrals (see Proposition 1.13). The dual space \hat{A} separates points of A by definition (because the set of left integrals is assumed to be faithful).

In this section, we will show that the space \hat{A} carries a product (dual to the coproduct on A) and a coproduct $\hat{\Delta}$ (dual to the product on A) so that the pair $(\hat{A}, \hat{\Delta})$ is again a regular weak multiplier Hopf algebra. As expected, the antipode \hat{S} on \hat{A} will simply be the adjoint of S . We will give a formula for the dual canonical multiplier \hat{E} . We will also show that this dual weak multiplier Hopf algebra again has a faithful set of (left) integrals and we will provide formulas for those. So, the pair $(\hat{A}, \hat{\Delta})$ is again an algebraic quantum groupoid, called the dual of (A, Δ) . The dual of $(\hat{A}, \hat{\Delta})$ is canonically isomorphic with the original weak multiplier Hopf algebra (A, Δ) .

In the case where there is a single faithful (left) integral, it is expected that \hat{A} has a single faithful integral as well. In that case it should be possible to give explicit formulas for the dual objects $\hat{\delta}, \hat{\sigma}, \dots$ in terms of the objects of the original weak multiplier Hopf algebras. These formulas are expected to be very similar (if not completely the same) as in the case of algebraic quantum groups (and algebraic quantum hypergroups). We refer to Section 4 for more comments on this case.

The dual algebra \hat{A}

We begin with the definition of the product in \hat{A} .

2.1. Proposition For any pair ω, ω' of elements in \hat{A} , we can define the product $\omega\omega' \in \hat{A}$ by

$$(\omega\omega')(x) = (\omega \otimes \omega')(\Delta(x))$$

where $x \in A$. If $\omega' = \varphi(a \cdot)$, where φ is a left integral and $a \in A$, then $\omega\omega' = \varphi(b \cdot)$ where now

$$b = ((\omega \circ S) \otimes \iota)\Delta(a).$$

Proof: The proof is standard. First we must argue that this product is well-defined as a linear functional on A . Take $\omega' = \sum_i \varphi_i(\cdot a_i)$ where φ_i are left integrals and $a_i \in A$. Then we see that

$$(\omega \otimes \omega')\Delta(x) = \sum_i (\omega \otimes \varphi_i)(\Delta(x)(1 \otimes a_i))$$

and this is defined for all x because for all a_i we have that $\Delta(x)(1 \otimes a_i)$ belongs to $A \otimes A$. One must also argue that $\omega\omega'$ is well-defined in the sense that it does not depend on the choice of the representation of ω' . For doing this, one assumes that ω is chosen as $\varphi(\cdot a)$ for a left integral φ and an element $a \in A$.

Next one has to show that the resulting element $\omega\omega'$ again belongs to \hat{A} . To see this, take $\omega' = \varphi(a \cdot)$ where φ is a left integral and $a \in A$. Using the formula in Proposition 1.5, we find

$$(\omega \otimes \omega')\Delta(x) = (\omega \otimes \varphi)((1 \otimes a)\Delta(x)) = ((\omega \circ S) \otimes \varphi)(\Delta(a)(1 \otimes x))$$

and we see that $\omega\omega' = \varphi(b \cdot)$ with $b = ((\omega \circ S) \otimes \iota)\Delta(a)$. This element b belongs to A as also $\omega \in \hat{A}$.

This completes the proof. \square

Also the proof of the following result is not so difficult.

2.2. Proposition The product defined on \hat{A} in the previous proposition makes \hat{A} into an associative algebra with a non-degenerate product. The algebra is also idempotent.

Proof: The *associativity* of the product on \hat{A} follows of course from the coassociativity of the coproduct Δ on A . In order to do this in a careful way, using the notion of coassociativity as defined in Definition 1.1 of [28], one can use the various representations of elements in \hat{A} as obtained in Proposition 1.13 of Section 1.

Next we show that the product is *non-degenerate*. Assume first that $\omega \in \hat{A}$ and that $\omega\omega' = 0$ for all $\omega' \in \hat{A}$. This implies that $\omega'((\omega \otimes \iota)\Delta(x)) = 0$ for all $x \in A$ and for all $\omega' \in \hat{A}$. Recall that $(\omega \otimes \iota)\Delta(x)$ is in A for all $x \in A$ because $\omega \in \hat{A}$. As \hat{A} is separating points of A , it follows that $(\omega \otimes \iota)\Delta(x) = 0$ for all $x \in A$. If we apply the counit, we find that $\omega(x) = 0$ for all x and this means that $\omega = 0$. Similarly, we will get $\omega = 0$ if $\omega\omega' = 0$ for all $\omega' \in \hat{A}$.

Finally we prove that the algebra is *idempotent*. To show this, let $\omega \in \hat{A}$, $a \in A$ and let φ be any left integral. Denote $\omega' = \varphi(a \cdot)$. As in the proof of Proposition 2.1, we find $\omega\omega' = \varphi(b \cdot)$ with $b = ((\omega \circ S) \otimes \iota)\Delta(a)$. Now we claim that any element in A is a linear combination of elements b of this form. This will imply that any element in \hat{A} can be written as a sum of products of elements in \hat{A} .

To prove the claim, assume that this is not the case. Then there exists a linear functional ρ on A so that $((\omega \circ S) \otimes \rho)\Delta(a) = 0$ for all $\omega \in \hat{A}$ and all $a \in A$. We now proceed as above (when proving that the product is non-degenerate), but we need to be a little more careful. Replace ω by $\omega(\cdot b)$ with $b \in A$ and use that \hat{A} separates points. This will give that $(S \otimes \rho)(\Delta(a))(b \otimes 1) = 0$ for all a, b . This means that $(\iota \otimes \rho)\Delta(a) = 0$ in $M(A)$ for all a . Using that the coproduct is full will give that $\rho = 0$. This completes the proof. \square

Remark that in the second part of the proof above, in stead of using the counit, we can also proceed by using that the coproduct is full (as we had to do in the last part of the proof where we could not apply the counit).

Also observe that for proving that \hat{A} is idempotent, we do not need that the original algebra A is idempotent.

The proofs of the previous results on the dual algebra are completely the same as in the case of multiplier Hopf algebras with integrals. See e.g. [23]. The construction of the coproduct is however, just as in the case of multiplier Hopf algebras, a little more subtle.

The pairing $\langle A, \hat{A} \rangle$

Before we investigate the coproduct on \hat{A} , let us first have a closer look at the non-degenerate pairing between the algebras A and \hat{A} . As an application, we will get a characterization of $M(\hat{A})$ that is similar as for algebraic quantum (hyper)groups, (see e.g. Proposition 3.4 in [5]).

In what follows, we will also use the *pairing notation*. So we will write $\langle a, \omega \rangle$ for $\omega(a)$ when $a \in A$ and $\omega \in \widehat{A}$. In this case, we will also use B for \widehat{A} and a, a', \dots for elements in A and b, b', \dots for elements in B .

Left and right multiplication in the algebras on either side, induce right and left actions as we see in the following proposition.

2.3. Proposition There exist left and right actions of A on B , as well as right and left actions of B on A given by the following formulas:

$$\begin{aligned} \langle a'a, b \rangle &= \langle a', a \triangleright b \rangle & \langle a, bb' \rangle &= \langle a \triangleleft b, b' \rangle \\ \langle aa', b \rangle &= \langle a', b \triangleleft a \rangle & \langle a, b'b \rangle &= \langle b \triangleright a, b' \rangle \end{aligned}$$

where $a, a' \in A$ and $b, b' \in B$. In the two cases, the left and right actions commute. Moreover, all these four actions are unital and non-degenerate.

Proof: First consider the *left action of A on B* . Assume that $\omega = \varphi(\cdot a')$. Then we have $a \triangleright \omega = \omega(\cdot a) = \varphi(\cdot aa')$ and this will again belong to \widehat{A} . We also see that $A \triangleright B = A$ because $A^2 = A$. So the action is unital. Finally, to show that the action is non-degenerate, assume that $\omega \in \widehat{A}$ and that $\omega(\cdot a) = 0$ for all a . Again because $A^2 = A$, it follows that $\omega = 0$. The arguments for the right action of A on B are completely similar. It is clear that the left action commutes with the right action.

Next consider the *right action of B on A* . We have that $a \triangleleft \omega = \sum_{(a)} \omega(a_{(1)})a_{(2)}$ where $\omega \in \widehat{A}$. We know that this belongs to A . We get indeed a right action of \widehat{A} on A . Similarly, we get the left action. They clearly commute.

To show that also this right action is unital, we have to prove that any element in A is the span of such elements. Suppose that this is not the case. Then, there exists a non-zero linear functional ρ on A such that $(\omega \otimes \rho)\Delta(a) = 0$ for all $a \in A$ and all $\omega \in \widehat{A}$. Then, as in the proof of Proposition 2.2, it follows that $\rho = 0$ and this gives a contradiction. Finally, also this action is non-degenerate because the pairing is non-degenerate and B is also idempotent (see Proposition 2.2). Similarly for the left action. \square

Observe that we know the existence of local units for A (see Proposition 4.8 in [28]). Then any unital action of A is non-degenerate. However, at this moment, we do not yet know that also \widehat{A} has local units. This will only follow when we have shown that it is also a regular weak multiplier Hopf algebra. This means that, at this level, we have to show that these actions are unital *and* non-degenerate.

Next, we will use the results of Proposition 2.3 to extend the pairing from $A \times B$ to $A \times M(B)$ and to $M(A) \times B$. The two cases are treated in a different way, just above. Remark e.g. that extending the pairing to $M(A) \times B$ is a special case of extending a *reduced* linear functional on A to $M(A)$. It is known that this can be done by using that A has local units. The other case is different.

First we have the easier case.

2.4. Proposition There is a unique extension of the pairing of A with \widehat{A} to a pairing of $M(A)$ with \widehat{A} , satisfying

$$\langle m, b \triangleleft a \rangle = \langle am, b \rangle \quad \text{and} \quad \langle m, a \triangleright b \rangle = \langle ma, b \rangle$$

whenever $a \in A$, $b \in B$ and $m \in M(A)$.

Proof: The proof is standard. Start with an element $m \in M(A)$ and $b \in B$. We use e.g. the fact that the right action \triangleleft of A on B is unital to write $b = \sum_i b_i \triangleleft a_i$. We try to define

$$\langle m, b \rangle = \sum_i \langle a_i m, b_i \rangle,$$

but we first have to show that this is well-defined. For this assume that $\sum_i b_i \triangleleft a_i = 0$. Take e in A so that $a_i e = a_i$ for all i . Then

$$\sum_i \langle a_i m, b_i \rangle = \sum_i \langle a_i e m, b_i \rangle = \sum_i \langle e m, b_i \triangleleft a_i \rangle = 0.$$

This proves that we can define the pairing, using the first formula. To prove that the second formula also holds, one replaces b by $\sum b_i \triangleleft a_i$, uses the first formula and again that the right action is unital. Uniqueness is obvious. \square

We now look at the other extension. Remark again that we used local units above and so, we have to find another way to prove this case.

2.5. Proposition The pairing on $A \times B$ can be extended to $A \times M(B)$ in such a way that

$$\langle a, bm \rangle = \langle a \triangleleft b, m \rangle \quad \text{and} \quad \langle a, mb \rangle = \langle b \triangleright a, m \rangle$$

whenever $a \in A$, $b \in B$ and $m \in M(B)$.

Proof: We start as in the previous proof. Now take $m \in M(B)$, $a \in A$, use the fact that the right action \triangleleft of B on A is unital and write $a = \sum_i a_i \triangleleft b_i$. We again want to define

$$\langle a, m \rangle = \sum_i \langle a_i, b_i m \rangle.$$

So, we assume that $\sum_i a_i \triangleleft b_i = 0$. Then we have, for all $b \in B$,

$$\begin{aligned} \sum_i \langle a_i \triangleleft (b_i m), b \rangle &= \sum_i \langle a_i, b_i m b \rangle \\ &= \sum_i \langle a_i \triangleleft b_i, m b \rangle = 0. \end{aligned}$$

This implies that $\sum_i a_i \triangleleft (b_i m) = 0$. If we apply the counit ε of A we find precisely that

$$\sum_i \langle a_i, b_i m \rangle = 0.$$

This shows that we can define the extension and that it will satisfy the first formula. An argument, similar as in the previous proof will give also the second formula. And again, uniqueness is trivial. \square

So, instead of using the property of having local units, we use the counit. Of course, the counit is the identity in $M(\widehat{A})$ and so, in a sense, we use that we already have the extension of the pairing to elements of the form $(a, 1)$ in $A \times \widehat{A}$. Indeed, we are using that $\varepsilon(a \triangleleft b) = \langle a, b \rangle$. But this can be justified by replacing b by an element of the form $a' \triangleright b$.

We will use this result later, in Proposition 2.10 below, to define the counit $\widehat{\varepsilon}$ on \widehat{A} by $\widehat{\varepsilon}(\omega) = \omega(1)$ for $\omega \in \widehat{A}$.

Recall that in the general case (just as for multiplier Hopf algebras), the pairing cannot be extended further to $M(A) \times M(\widehat{A})$.

Before we continue with the study of the coproduct on \widehat{A} , we apply the above results to get the following characterization of elements in $M(\widehat{A})$. We have the same characterization as for algebraic quantum (hyper-)groups, see e.g. Proposition 3.4 in [5].

2.6. Proposition Let ω be a linear functional in A' . Then there is a multiplier m in $M(\widehat{A})$ so that $\omega(a) = \langle a, m \rangle$ if and only if

$$(\omega \otimes \iota)\Delta(a) \quad \text{and} \quad (\iota \otimes \omega)\Delta(a)$$

are in A for all $a \in A$.

Proof: i) First let $m \in M(\widehat{A})$ and define ω in A' by $\omega(a) = \langle a, m \rangle$. We use the extended pairing as obtained in the previous proposition. Now we use e.g. that the left action of \widehat{A} on A is unital. Then we can write any element $a \in A$ as $\sum_i b_i \triangleright a_i$. This will give

$$(\iota \otimes \omega)\Delta(a) = \sum_i (mb_i) \triangleright a_i$$

and this belongs to A . Similarly, if $a = \sum_i b_i \triangleleft a_i$, we find

$$(\omega \otimes \iota)\Delta(a) = \sum_i a_i \triangleleft (b_i m)$$

and we have that also $(\omega \otimes \iota)\Delta(a)$ belongs to A .

ii) Conversely, assume that we have an element ω in A' satisfying the conditions in the formulation of the proposition. Then we can define a multiplier m of \widehat{A} by

$$(m\omega')(a) = (\omega \otimes \omega')\Delta(a) \quad \text{and} \quad (\omega' m)(a) = (\omega' \otimes \omega)\Delta(a)$$

where $\omega' \in \widehat{A}$. Arguments as used earlier (see the proof of Proposition 2.1) will be needed in order to have that $m\omega'$ and $\omega' m$ again belong to \widehat{A} . The rest of the argument is more or less trivial. \square

The proof is essentially the same as for an algebraic quantum group and even an algebraic quantum hypergroup. See e.g. Proposition 3.4 in [5].

The coproduct $\widehat{\Delta}$ on \widehat{A}

Now, we construct the coproduct. We do it by first defining the canonical maps \widehat{T}_1 and \widehat{T}_2 in the following proposition. As before, we use the pairing and we write $\langle a, \omega \rangle$ for $\omega(a)$ when $a \in A$ and $\omega \in \widehat{A}$ (or even in A'). Similarly, we use this notation when $a \in A \otimes A$ and ω in $\widehat{A} \otimes \widehat{A}$ (or even $\omega \in (A \otimes A)'$).

2.7. Proposition There is a regular coproduct $\widehat{\Delta}$ on \widehat{A} so that the associated canonical maps \widehat{T}_1 and \widehat{T}_2 , satisfy

$$\langle x \otimes y, \widehat{T}_1(\omega \otimes \omega') \rangle = \langle T_2(x \otimes y), \omega \otimes \omega' \rangle \quad (2.1)$$

$$\langle x \otimes y, \widehat{T}_2(\omega \otimes \omega') \rangle = \langle T_1(x \otimes y), \omega \otimes \omega' \rangle \quad (2.2)$$

for all $x, y \in A$ and $\omega, \omega' \in \widehat{A}$.

Proof: i) The formulas (2.1) and (2.2) in the formulation of the proposition define \widehat{T}_1 and \widehat{T}_2 as linear maps from $\widehat{A} \otimes \widehat{A}$ to the linear dual space $(A \otimes A)'$ of $A \otimes A$. We claim that these maps have range in $\widehat{A} \otimes \widehat{A}$.

To show this for \widehat{T}_1 , we take $\omega' = \varphi(a \cdot)$ where φ is a left integral on A and where $a \in A$. Then we have for all $x \in A$ and $y \in A$ that

$$\begin{aligned} \langle T_2(x \otimes y), \omega \otimes \omega' \rangle &= (\omega \otimes \varphi)((x \otimes a)\Delta(y)) \\ &= (\omega \otimes \varphi)((x \otimes 1)((S \otimes \iota)(\Delta(a)(1 \otimes y)))). \end{aligned}$$

We see that

$$\widehat{T}_1(\omega \otimes \omega') = \sum_{(a)} \omega(\cdot S(a_{(1)})) \otimes \varphi(a_{(2)} \cdot) \quad (2.3)$$

and so $\widehat{T}_1(\omega \otimes \omega') \in \widehat{A} \otimes \widehat{A}$ because $((S \otimes \iota)\Delta(a))(a' \otimes 1) \in A \otimes A$ when $a, a' \in A$. To prove that \widehat{T}_2 maps $\widehat{A} \otimes \widehat{A}$, we use appropriate representations of the elements ω, ω' of \widehat{A} , using right integrals.

ii) Next, we show that there is a coproduct $\widehat{\Delta}$ on \widehat{A} so that \widehat{T}_1 and \widehat{T}_2 are the associated canonical maps. For this we need that

$$(\omega \otimes 1)(\widehat{T}_1(\omega' \otimes \omega'')) = (\widehat{T}_2(\omega \otimes \omega'))(1 \otimes \omega'') \quad (2.4)$$

for all ω, ω' and ω'' in \widehat{A} . Remark that this equation makes sense in $\widehat{A} \otimes \widehat{A}$. And indeed, it will imply that there is a map

$$\widehat{\Delta} : \widehat{A} \rightarrow M(\widehat{A} \otimes \widehat{A})$$

so that $\widehat{T}_1(\omega' \otimes \omega'') = \Delta(\omega')(1 \otimes \omega'')$ and $\widehat{T}_2(\omega \otimes \omega') = (\omega \otimes 1)\widehat{\Delta}(\omega')$ for all ω, ω' and ω'' in \widehat{A} .

To prove that (2.4) holds, one can verify that

$$\begin{aligned} \langle x \otimes y, (\omega \otimes 1)(\widehat{T}_1(\omega' \otimes \omega'')) \rangle &= \langle \Delta(x) \otimes y, (\iota \otimes \widehat{T}_1)(\omega \otimes \omega' \otimes \omega'') \rangle \\ &= \langle (\iota \otimes T_2)(\Delta(x) \otimes y), \omega \otimes \omega' \otimes \omega'' \rangle \\ &= \langle (\Delta(x) \otimes 1)(1 \otimes \Delta(y)), \omega \otimes \omega' \otimes \omega'' \rangle \end{aligned}$$

for all $x, y \in A$ and ω, ω' and ω'' in \widehat{A} . Similarly also

$$\langle x \otimes y, (\widehat{T}_2(\omega \otimes \omega'))(1 \otimes \omega'') \rangle = \langle (\Delta(x) \otimes 1)(1 \otimes \Delta(y)), \omega \otimes \omega' \otimes \omega'' \rangle$$

for all such elements. Some care is needed, but remark that in all these equations, the legs of $\Delta(x)$ and the legs of $\Delta(y)$ are well covered by using the appropriate forms of the elements in \widehat{A} .

iii) That this coproduct is regular, is proven as in i), using again well-chosen representations of the elements in \widehat{A} .

iv) Next we argue that $\widehat{\Delta}$ is a homomorphism. We will do this with the pairing notation and therefore, we will again use B for \widehat{A} and denote the coproduct $\widehat{\Delta}$ on B simply by Δ . We will also use the Sweedler notation. The reader can verify that the necessary coverings exist (possibly also induced by the pairing and *repeated* coverings).

Take a, a' in A and b, b', c in B . Then we have

$$\begin{aligned} \langle a \otimes a', \Delta(bb')(1 \otimes c) \rangle &= \langle (a \otimes 1) \Delta(a'), bb' \otimes c \rangle \\ &= \sum_{(a')} \langle aa'_{(1)}, bb' \rangle \langle a'_{(2)}, c \rangle \\ &= \sum_{(a')} \langle \Delta(aa'_{(1)}), b \otimes b' \rangle \langle a'_{(2)}, c \rangle \\ &= \sum_{(a')} \langle \Delta(aa'_{(1)}) \otimes a'_{(2)}, b \otimes b' \otimes c \rangle. \end{aligned}$$

We have used that the map \widehat{T}_1 is the adjoint of T_2 . Also remark that c will provide the necessary coverings.

On the other hand we find

$$\begin{aligned} \langle a \otimes a', \Delta(b) \Delta(b')(1 \otimes c) \rangle &= \sum_{(b), (b')} \langle a \otimes a', b_{(1)} b'_{(1)} \otimes b_{(2)} b'_{(2)} c \rangle \\ &= \sum_{(b), (b')} \langle \Delta(a) \otimes a', b_{(1)} \otimes b'_{(1)} \otimes b_{(2)} b'_{(2)} c \rangle \\ &= \sum_{(b')} \langle (\Delta(a) \otimes 1) \Delta_{13}(a'), b \otimes b'_{(1)} \otimes b'_{(2)} c \rangle \\ &= \langle (\Delta(a) \otimes 1) \Delta^{(2)}(a'), b \otimes b' \otimes c \rangle. \end{aligned}$$

Again we have used that \widehat{T}_1 is the adjoint of T_2 , now two times. And again, all the time, c will take care of the necessary coverings.

We see that both expressions are the same and this proves that Δ is a homomorphism on B .

v)) It remains to be shown that $\widehat{\Delta}$ is coassociative. For this we need to show that the maps $\iota \otimes \widehat{T}_1$ and $\widehat{T}_2 \otimes \iota$ commute on $\widehat{A} \otimes \widehat{A}$. This will follow by duality if we can show that the maps $\iota \otimes T_2$ and $T_1 \otimes \iota$ commute on $A \otimes A$. And this in turn should follow from the *associativity* of the product in A . Indeed, we get for all x, y, z in A that

$$\begin{aligned} (T_1 \otimes \iota)(\iota \otimes T_2)(x \otimes y \otimes z) &= (T_1 \otimes \iota)(x \otimes (y \otimes 1) \Delta(z)) \\ &= (\Delta(x) \otimes 1)(1 \otimes y \otimes 1)(1 \otimes \Delta(z)) \end{aligned}$$

and similarly for $(\iota \otimes T_2)(T_1 \otimes \iota)$. □

Also this proof is essentially the same as for regular multiplier Hopf algebras with integrals. Another similar treatment is found in Section 3 of [5] in the case of algebraic quantum hypergroups.

Observe in the last part of the proof, that the dual maps $\iota \otimes \widehat{T_1}$ and $\widehat{T_2} \otimes \iota$ commute, is equivalent with the fact that the maps $\iota \otimes T_2$ and $T_1 \otimes \iota$ commute on $A \otimes A$. This last fact is different from the property that the maps $\iota \otimes T_1$ and $T_2 \otimes \iota$ commute on $A \otimes A$. The first commutation rule is equivalent with the associativity of the product in A , whereas the latter with the *coassociativity of the coproduct* on A .

We finish this item on the coproduct $\widehat{\Delta}$ by proving the following expected formula. Observe that we need the extension of the pairing from $(A \otimes A) \times (\widehat{A} \otimes \widehat{A})$ to $(A \otimes A) \times M(\widehat{A} \otimes \widehat{A})$. This extension can be obtained as in Proposition 2.5. We again use the pairing conventions.

2.8. Proposition For all a, a' in A and $b \in B$, we have $\langle aa', b \rangle = \langle a \otimes a', \Delta(b) \rangle$.

Proof: Take $a, a' \in A$ and $b, b' \in B$. Then we have

$$\begin{aligned} \langle a \otimes (b' \triangleright a'), \Delta(b) \rangle &= \langle a \otimes a', \Delta(b)(1 \otimes b') \rangle \\ &= \langle (a \otimes 1) \Delta(a'), b \otimes b' \rangle \\ &= \langle a(b' \triangleright a'), b \rangle. \end{aligned}$$

Now the result follows because the left action of B on A is unital.

Existence of a counit and fullness of the coproduct

Next, we show that there exists a counit $\widehat{\varepsilon}$ and that the coproduct $\widehat{\Delta}$ is full (so that hence the counit is unique). This is what we show first.

2.9. Proposition The coproduct $\widehat{\Delta}$ on \widehat{A} , as obtained in Proposition 2.7, is full.

Proof: We have seen in the proof of the Proposition 2.7 that

$$\widehat{\Delta}(\omega)(1 \otimes \omega') = \sum_{(a)} \omega(\cdot S(a_{(1)})) \otimes \varphi(a_{(2)} \cdot)$$

if $\omega, \omega' \in \widehat{A}$ and if ω' has the form $\varphi(a \cdot)$ for $a \in A$ and a left integral φ on A . See Equation (2.3). If γ is the linear functional on \widehat{A} , given by the evaluation in a point c of A , we find that

$$(\iota \otimes \gamma(\cdot \omega')) \widehat{\Delta}(\omega) = \sum_{(a)} \omega(\cdot S(a_{(1)})) \varphi(a_{(2)} c).$$

So, in order to show that the 'left leg' of $\widehat{\Delta}$ is all of \widehat{A} , we must have that any element in A is a linear combination of elements of the form

$$\sum_{(a)} S(a_{(1)}) \varphi(a_{(2)} c)$$

where $a, c \in A$ and where φ is a left integral on A . Assume that this is not true. Then there exists a linear functional ρ on A so that

$$\sum_{(a)} \rho(S(a'_{(1)})) \varphi(a_{(2)} c) = 0$$

for all $a', c \in A$ and all left integrals φ . Because we assume that there is a faithful set of left integrals on A , we then have that

$$\sum_{(a)} \rho(S(a_{(1)}))a_{(2)} = 0$$

for all a' . Because the original coproduct is assume to be full, this implies that $\rho = 0$. This proves that the left leg of $\hat{\Delta}$ is all of \hat{A} . Similarly for the right leg. \square

2.10. Proposition There is a unique counit $\hat{\varepsilon}$ on \hat{A} given by $\hat{\varepsilon}(\omega) = \omega(1)$.

Proof: First remark that we can use Proposition 2.4 (with $m = 1$ in $M(A)$) in order to define $\omega(1)$ for $\omega \in \hat{A}$. It is also possible to consider $\varphi(a)$ when $\omega = \varphi(\cdot a)$ but then one would need to show that this is a well-defined map from \hat{A} to \mathbb{C} . The proof of this can be given by using local units for A , just as we needed for the proof of Proposition 2.4.

By definition, we will have $\hat{\varepsilon}(\omega(\cdot a)) = \omega(a)$ as well as $\hat{\varepsilon}(\omega(a \cdot)) = \omega(a)$ for all $a \in A$ and $\omega \in \hat{A}$.

Consider ω and ω' in \hat{A} . We know by definition that

$$(\hat{\Delta}(\omega)(1 \otimes \omega'))(a \otimes a') = (\omega \otimes \omega')((a \otimes 1)\Delta(a')).$$

Therefore we find that

$$(\hat{\varepsilon} \otimes \iota)(\hat{\Delta}(\omega)(1 \otimes \omega')) = (\omega \otimes \omega') \circ \Delta = \omega\omega'.$$

Similarly we will get

$$(\iota \otimes \hat{\varepsilon})((\omega \otimes 1)\hat{\Delta}(\omega')) = \omega\omega'.$$

This proves that $\hat{\varepsilon}$ is a counit. And because we know already that the coproduct is full, it is the unique coproduct on the pair $(\hat{A}, \hat{\Delta})$. \square

The generalized inverses of the dual canonical maps

We now look for the appropriate generalized inverses \hat{R}_1 and \hat{R}_2 of \hat{T}_1 and \hat{T}_2 respectively. Because we have that \hat{T}_1 and \hat{T}_2 are adjoint to T_2 and T_1 , we expect that \hat{R}_1 and \hat{R}_2 are adjoint to R_2 and R_1 respectively. Of course, we also expect that the antipode \hat{S} of the pair $(\hat{A}, \hat{\Delta})$ will be the adjoint of S .

First we consider the adjoint of S .

2.11. Proposition There is a map $\hat{S} : \hat{A} \rightarrow \hat{A}$, satisfying

$$\langle a, \hat{S}(b) \rangle = \langle S(a), b \rangle$$

for all $a \in A$ and $b \in \hat{A}$. This map is also bijective.

Proof: The above formula can be used to define $\widehat{S}(b) \in A'$ for all $b \in A'$. We just have to argue that $\widehat{S}(b) \in \widehat{A}$ for all $b \in \widehat{A}$.

So take e.g. $b = \varphi(\cdot c)$ where $c \in A$ and φ is a left integral on A . Then

$$\langle a, \widehat{S}(b) \rangle = \varphi(S(a)c) = (\varphi \circ S)(S^{-1}(c)a)$$

for all a so that $\widehat{S}(b) = (\varphi \circ S)(S^{-1}(c) \cdot)$. We know that $\varphi \circ S$ is a right integral. From Proposition 1.13 we find that $\widehat{S}(b)$ again belongs to \widehat{A} .

We also see from the above argument that \widehat{S} is a bijective map from \widehat{A} to itself. \square

As the map S is both an anti-algebra and an anti-coalgebra map, the same will be true for its adjoint (because the product and the coproduct are dual to each other). This is standard.

Remark that at this moment, we still have not shown that the pair $(\widehat{A}, \widehat{\Delta})$ is again a weak multiplier Hopf algebra and so we cannot yet argue that the map \widehat{S} , defined above, is actually the antipode of $(\widehat{A}, \widehat{\Delta})$. This will become obvious later.

A similar remark applies for the maps \widehat{R}_1 and \widehat{R}_2 we consider next.

2.12. Proposition There exist maps \widehat{R}_1 and \widehat{R}_2 from $\widehat{A} \otimes \widehat{A}$ to itself, satisfying

$$\langle a \otimes a', \widehat{R}_1(b \otimes b') \rangle = \langle R_2(a \otimes a'), b \otimes b' \rangle \quad (2.5)$$

$$\langle a \otimes a', \widehat{R}_2(b \otimes b') \rangle = \langle R_1(a \otimes a'), b \otimes b' \rangle \quad (2.6)$$

for all $a, a' \in A$ and $b, b' \in B$. Moreover we have

$$\widehat{R}_1(b \otimes b') = \sum_{(b)} b_{(1)} \otimes \widehat{S}(b_{(2)})b' \quad (2.7)$$

$$\widehat{R}_2(b \otimes b') = \sum_{(b')} b \widehat{S}(b'_{(1)}) \otimes b'_{(2)} \quad (2.8)$$

for all b, b' . The maps \widehat{R}_1 and \widehat{R}_2 are generalized inverses of \widehat{T}_1 and \widehat{T}_2 respectively.

Proof: Again we can use the formulas (2.5) and (2.6) above to define $\widehat{R}_1(b \otimes b')$ and $\widehat{R}_2(b \otimes b')$ in $(A \otimes A)'$ for $b, b' \in A'$. It can be shown as in the proof of Proposition 2.7 that actually these elements belong to $\widehat{A} \otimes \widehat{A}$ if $b, b' \in \widehat{A}$. In fact, it is also possible to use that \widehat{S} is an anti-automorphism of the algebra \widehat{A} , combined with the fact that the canonical maps on $\widehat{A} \otimes \widehat{A}$ have range in $\widehat{A} \otimes \widehat{A}$. This takes care of the first statement of the proposition.

Let us now prove Equation (2.7). The proof of (2.8) is completely similar.

Take $a, a' \in A$ and $b, b' \in \widehat{A}$. If we consider the map \widehat{R}_1 as in the formula (2.7), we get

$$\begin{aligned} \langle a \otimes a', \widehat{R}_1(b \otimes b') \rangle &= \langle a \otimes \Delta(a'), (\iota \otimes \widehat{S} \otimes \iota)(\widehat{\Delta}(b) \otimes b') \rangle \\ &= \langle a \otimes (S \otimes \iota)\Delta(a'), \widehat{\Delta}(b) \otimes b' \rangle \\ &= \langle (a \otimes 1)(S \otimes \iota)\Delta(a'), b \otimes b' \rangle \\ &= \langle R_2(a \otimes a'), b \otimes b' \rangle. \end{aligned}$$

This proves the equality (2.7).

Finally, it easily follows that \widehat{R}_1 is a generalized inverse of \widehat{T}_1 because they are adjoints of R_2 and T_2 that themselves are generalized inverses of each other. Similarly \widehat{R}_2 is a generalized inverse of \widehat{T}_2 . This completes the proof. \square

One can check that the necessary coverings can be found for the formulas above.

Also observe, as we already have the counit $\widehat{\varepsilon}$ available, that it follows that the map \widehat{S} satisfies the expected formulas

$$\sum_{(b)} b_{(1)} \widehat{S}(b_{(2)}) b_{(3)} = b \quad \text{and} \quad \sum_{(b)} \widehat{S}(b_{(1)}) b_{(2)} \widehat{S}(b_{(3)}) = \widehat{S}(b)$$

for all b .

For convenience in what follows, we will no longer write the $\widehat{\Delta}$, but simply use also Δ for the dual coproduct. Similarly for the dual canonical maps and there generalized inverses.

The canonical idempotent \widehat{E}

It is not hard to guess what the canonical idempotent \widehat{E} should be. Indeed, eventually it has to be $\Delta(1)$ and so we expect that \widehat{E} , as an element in the dual space $(A \otimes A)'$, should satisfy

$$\langle a \otimes a', \widehat{E} \rangle = \langle aa', 1 \rangle = \varepsilon(aa')$$

We will now use this formula to *define* \widehat{E} and prove its properties. In particular, we show that this is indeed the canonical idempotent for the dual $(\widehat{A}, \widehat{\Delta})$.

2.13. Proposition There is a multiplier \widehat{E} in $M(B \otimes B)$ defined by

$$\langle a \otimes a', \widehat{E} \rangle = \varepsilon(aa')$$

for all $a, a' \in A$. It satisfies

$$\widehat{E}(b \otimes b') = T_1 R_1(b \otimes b') \quad \text{and} \quad (b \otimes b') \widehat{E} = T_2 R_2(b \otimes b')$$

for all $b, b' \in B$. In particular, it is an idempotent multiplier determining the ranges of the canonical maps as in the definition of a weak multiplier Hopf algebra (Definition 1.14 in [28]).

Proof: We first define \widehat{E} in $(A \otimes A)'$ using the above formula. Next take two elements b, b' in \widehat{A} . Then, using the product in $(A \otimes A)'$, we get

$$\langle a \otimes a', \widehat{E}(b \otimes b') \rangle = \sum_{(a)(a')} \varepsilon(a_{(1)} a'_{(1)}) \langle a_{(2)}, b \rangle \langle a'_{(2)}, b' \rangle. \quad (2.9)$$

Now we have

$$\sum_{(a')} \Delta(a)(a'_{(1)} \otimes 1) \otimes a'_{(2)} = \sum_{(a')} \Delta(aa'_{(1)})(1 \otimes S(a'_{(2)})) \otimes a'_{(3)}$$

and if we apply ε on the first factor, we find

$$\sum_{(a),(a')} \varepsilon(a_{(1)}a'_{(1)})a_{(2)} \otimes a'_{(2)} = \sum_{(a')} aa'_{(1)}S(a'_{(2)}) \otimes a'_{(3)} = R_2T_2(a \otimes a').$$

If we use this in the formula (2.9), we find

$$\langle a \otimes a', \widehat{E}(b \otimes b') \rangle = \langle R_2T_2(a \otimes a'), b \otimes b' \rangle$$

for all a, a' and b, b' . This shows that $\widehat{E}(b \otimes b') = T_1R_1(b \otimes b')$. In a similar way, we find that also $(b \otimes b')\widehat{E} = T_2R_2(b \otimes b')$. \square

Next we show that the coproduct has the right behavior on the legs of the canonical multiplier \widehat{E} .

2.14. Proposition We have

$$\begin{aligned} (\Delta \otimes \iota)\widehat{E} &= (\widehat{E} \otimes 1)(1 \otimes \widehat{E}) \\ (\Delta \otimes \iota)\widehat{E} &= (1 \otimes \widehat{E})(\widehat{E} \otimes 1). \end{aligned}$$

Proof: To prove the first equality, multiply from the left with $\Delta(b) \otimes b'$ where $b, b' \in B$. Then we see that it is equivalent with showing that

$$(\Delta \otimes \iota)T_2R_2 = (\iota \otimes T_2R_2)(\Delta \otimes \iota).$$

This equation is adjoint to the equation

$$R_1T_1(aa' \otimes a'') = (a \otimes 1)R_1T_1(a' \otimes a'')$$

for $a, a', a'' \in A$. And because $R_1T_1(a \otimes a') = (a \otimes 1)F_1(1 \otimes a')$ for all $a, a' \in A$ where $F_1 = (\iota \otimes S)E$, (see Propositions 4.5 and 4.7 in [28]), this equation is true.

To prove the other equality $(\Delta \otimes \iota)\widehat{E} = (1 \otimes \widehat{E})(\widehat{E} \otimes 1)$, we multiply from the right with $\Delta(b) \otimes b'$. Then the formula is equivalent with

$$(\Delta \otimes \iota)T_1R_1 = (\iota \otimes T_1R_1)(\Delta \otimes \iota).$$

This in turn is adjoint to the equation

$$R_2T_2(aa' \otimes a'') = (a \otimes 1)R_2T_2(a' \otimes a'')$$

for all a, a', a'' . And this is again true because $R_2T_2(a \otimes a') = (a \otimes 1)F_2(1 \otimes a')$ with $F_2 = (S \otimes \iota)E$, see again Propositions 4.5 and 4.7 in [28]. \square

Now we are ready for the main result.

2.15. Theorem The dual pair $(\widehat{A}, \widehat{\Delta})$ is again a regular weak multiplier Hopf algebra.

Proof: We will use Theorem 2.9 of [28].

i) We have shown in Proposition 2.2 that \widehat{A} is a non-degenerate, idempotent algebra. We also have proven in Proposition 2.7 and 2.9 that there is a regular and full

coproduct $\widehat{\Delta}$ on \widehat{A} . And finally, in Proposition 2.10, that this coproduct admits a counit.

ii) We have defined the linear map \widehat{S} from \widehat{A} to itself (in Proposition 2.11) and the associated maps \widehat{R}_1 and \widehat{R}_2 with the formulas (2.7) and (2.8) in Proposition 2.12. They are canonical inverses of \widehat{T}_1 and \widehat{T}_2 respectively. It follows that the map \widehat{S} satisfies the requirements in item i) and the equalities (2.5) in Theorem 2.9 of [28].

iii) Finally, we have found the dual canonical multiplier \widehat{E} satisfying the necessary formulas as proven in Proposition 2.13 and 2.14.

Hence all conditions of Theorem 2.9 in [28] are fulfilled and we have that the pair $(\widehat{A}, \widehat{\Delta})$ is a weak multiplier Hopf algebra. We know that \widehat{S} is its antipode and because this is a bijective map from \widehat{A} to itself (see 2.11), the weak multiplier Hopf algebra $(\widehat{A}, \widehat{\Delta})$ is regular.

This completes the proof. \square

Existence of integrals on the dual

Next we will show that the dual weak multiplier Hopf algebra also has a faithful set of integrals.

2.16. Proposition For every element $a \in A$, there exists a right invariant functional ψ_a on B such that

$$\psi_a(\omega) = \varphi(a\varepsilon_s(c)) \quad \text{when} \quad \omega = \varphi(\cdot c) \quad (2.10)$$

for any left integral φ on A and any element $c \in A$.

Proof: Fix an element $a \in A$.

i) We first show that we can define ψ_a such that (2.10) holds. To do this, assume that we have a finite number of left integrals φ_i and elements $c_i \in A$ so that

$$\sum_i \varphi_i(\cdot c_i) = 0.$$

This implies that

$$\sum_i (\iota \otimes \varphi_i)(\Delta(x)(1 \otimes c_i)) = 0$$

for all $x \in A$ and if we use one the formulas in Proposition 1.5, we see that also

$$\sum_i (\iota \otimes \varphi_i)((1 \otimes x)\Delta(c_i)) = 0$$

for all x . Then, using the Sweedler notation, we find

$$\sum_{i, (c_i)} aS(c_{i(1)}) \otimes \varphi_i(\cdot c_{i(2)}) = 0.$$

If we apply the evaluation map $x \otimes \omega \mapsto \omega(x)$, to this equation we get

$$\sum_{i, (c_i)} \varphi_i(aS(c_{i(1)})c_{i(2)}) = \sum_i \varphi_i(a\varepsilon_s(c_i)) = 0.$$

It follows that we can define a linear map ψ_a on B as in (2.10) above.

ii) Next we show that the functional ψ_a is right invariant on B . To do this, take a left integral φ on A and two elements $c, c' \in A$. Let $\omega = \varphi(\cdot c)$ as before. Then

$$\psi_a(\omega(\cdot c')) = \varphi(a\varepsilon_s(c'c)) = \varphi(a\varepsilon_s(\varepsilon_s(c')c))$$

where we have used that $\varepsilon_s(c'c) = \varepsilon_s(\varepsilon_s(c')c)$.

We see that

$$\psi_a(\omega(\cdot c')) = \psi_a(\omega(\cdot \varepsilon_s(c'))).$$

Using the Sweedler notation, the pairing notation and writing b for ω we get

$$\begin{aligned} \sum_{(b)} \psi_a(b_{(1)} \langle c', b_{(2)} \rangle) &= \sum_{(b)} \psi_a(b_{(1)} \langle \varepsilon_s(c'), b_{(2)} \rangle) \\ &= \sum_{(b)} \psi_a(b_{(1)} \langle c', \varepsilon_s(b_{(2)}) \rangle) \end{aligned}$$

where we have used that the source map is *self-dual*.

This exactly means now that

$$\sum_{(b)} \psi_a(b_{(1)} b_{(2)}) = \sum_{(b)} \psi_a(b_{(1)} \varepsilon_s(b_{(2)})),$$

proving the right invariance of ψ_a . □

In the proof above, we have used that the source and target maps are self-dual. On a formal level, this is easy to see. Indeed, given $a \in A$ and $b \in B$, we have

$$\begin{aligned} \langle a, \varepsilon_s(b) \rangle &= \sum_{(b)} \langle a, S(b_{(1)})b_{(2)} \rangle \\ &= \sum_{(a), (b)} \langle a_{(1)} \otimes a_{(2)}, S(b_{(1)}) \otimes b_{(2)} \rangle \\ &= \sum_{(a), (b)} \langle S(a_{(1)}) \otimes a_{(2)}, b_{(1)} \otimes b_{(2)} \rangle \\ &= \sum_{(a)} \langle S(a_{(1)})a_{(2)}, b \rangle = \langle \varepsilon_s(a), b \rangle. \end{aligned}$$

We see that ε_s is a self-dual map. Similarly we have

$$\langle a, \varepsilon_t(b) \rangle = \langle \varepsilon_t(a), b \rangle$$

for all $a \in A$ and $b \in B$.

The resulting formulas make sense by using the extensions of the pairings to the multiplier algebras on one side. However, the intermediate formulas in the argument are problematic because there is no obvious covering. The arguments can be made correct but the formulas get more involved.

In the next section we have an example where the coproduct on the dual actually maps into the algebraic tensor product and then, this property is more easy to obtain. We refer to [26] where this is explained in greater detail.

Next, we show that the set of right integrals is faithful.

2.17. Proposition The set of right integrals ψ_a with $a \in A$, as defined in the previous proposition, is faithful (in the sense of Definition 1.11).

Proof: i) Fix $\omega \in B$ and assume that $\psi_a(\omega\omega') = 0$ for all $a \in A$ and all $\omega' \in B$. We have to show that $\omega = 0$. If $\omega' = \varphi(\cdot c')$, then one can show that $\omega\omega' = \varphi(\cdot c)$ where $c = (\omega \circ S^{-1} \otimes \iota)\Delta(c')$. The argument is as in the proof of Proposition 2.1.

It follows that $\psi_a(\omega\omega') = \varphi(a\varepsilon_s(c))$. Now

$$\varepsilon_s(c) = \sum_{(c')} \omega(S^{-1}(c'_{(1)}))\varepsilon(c'_{(2)})$$

so that

$$\psi_a(\omega\omega') = \sum_{(c')} \varphi(a\varepsilon_s(c'_{(2)}))\omega(S^{-1}(c'_{(1)})).$$

Because we assume that this is 0 for all a and all φ , we see that

$$\sum_{(c')} \omega(S^{-1}(c'_{(1)}))\varepsilon_s(c'_{(2)}) = 0.$$

As this is true for all c' , we are allowed to multiply with $S(c'_{(3)})$ from the right and obtain that also

$$\sum_{(c')} \omega(S^{-1}(c'_{(1)}))S(c'_{(2)}) = 0$$

for all c' . Then we can apply the counit and we find that $\omega = 0$.

ii) Now fix $\omega' \in B$ and assume that $\psi_a(\omega\omega') = 0$ for all $a \in A$ and all $\omega' \in B$. We have to show that $\omega' = 0$. If we follow again the calculations as in i), we arrive at the fact that now

$$\sum_{(c')} \omega(S^{-1}(c'_{(1)}))S(c'_{(2)}) = 0$$

for all ω . This now implies that $c' = 0$ and hence $\omega' = 0$. \square

Now, we can prove the second main result.

2.18. Theorem The dual $(\widehat{A}, \widehat{\Delta})$ of the weak multiplier Hopf algebra (A, Δ) again has a faithful set of integrals. Then we can consider the dual of $(\widehat{A}, \widehat{\Delta})$ and it is canonically isomorphic with the original pair (A, Δ) .

Proof: Most of this has been shown. Indeed, in Theorem 2.15, we have shown that $(\widehat{A}, \widehat{\Delta})$ is again a regular weak multiplier Hopf algebra. On the other hand, in Proposition 2.17 we found that it has a faithful set of integrals. So, the only thing that is left is the existence of a bijection of the dual of \widehat{A} to A .

If we look again at the proof of Proposition 2.17 above, we see e.g. that $\psi_a(\cdot \omega')$ is given by evaluation in the point $\sum_{(c')} S^{-1}(c'_{(1)})\varphi(a\varepsilon_s(c'_{(2)}))$ where $\omega' = \varphi(\cdot c')$. We claim that these elements belong to A and that all elements of A are of this form.

First, recall that

$$\sum_{(c')} c'_{(1)} \otimes \varepsilon_s(c'_{(2)})b = R_1 T_1(c' \otimes b) = (c' \otimes 1)F_1(1 \otimes b) \quad (2.11)$$

for any $b \in A$. If we replace c' by cc' we see that

$$(cc' \otimes 1)F_1 = \sum_{(c')} cc'_{(1)} \otimes \varepsilon_s(c'_{(2)}) \quad (2.12)$$

and it follows that $\sum_{(c')} c'_{(1)} \otimes \varepsilon_s(c'_{(2)})$ is an element in $A \otimes M(A)$ and so $\sum_{(c')} c'_{(1)} \otimes a\varepsilon_s(c'_{(2)})$ is in $A \otimes A$. This proves the first claim.

To prove the second part of the claim, assume that ρ is a linear functional that vanishes on all such elements. By the fact that we have a faithful set of integrals on A , this will give that $(\rho(S^{-1}(\cdot)) \otimes \iota)((c' \otimes 1)F_1) = 0$ for all c' . This implies that $\rho = 0$ because $F_1 = (\iota \otimes S)E$ and E is full. \square

The source and target of the dual

We have seen that the source and target maps are self-dual maps. There is however, another important relation between the source and target on the dual and the source and target on the original weak multiplier Hopf algebra.

2.19. Proposition There is an isomorphism γ_s from the source algebra $\varepsilon_s(A)$ of A to the target algebra $\varepsilon_t(\widehat{A})$ of the dual, given by the formula

$$\langle ya, b \rangle = \langle a, b\gamma_s(y) \rangle \quad (2.13)$$

for all $a \in A$ and $b \in B$. Here $y \in \varepsilon_s(A)$. Similarly, there is an isomorphism γ_t from the target algebra $\varepsilon_t(A)$ of A to the source algebra $\varepsilon_s(\widehat{A})$ of the dual, given by the formula

$$\langle ax, b \rangle = \langle a, \gamma_t(x)b \rangle \quad (2.14)$$

for all $a \in A$ and $b \in B$. Here $x \in \varepsilon_t(A)$.

Proof: We use that $T_1 R_1(a \otimes a') = E(a \otimes a')$ for $a, a' \in A$ and that $R_2 T_2(b \otimes b') = (b \otimes 1)F_2(1 \otimes b')$ when $b, b' \in \widehat{A}$ where $F_2 = (S \otimes \iota)\widehat{E}$. We also know that the canonical map T_2 on $\widehat{A} \otimes \widehat{A}$ is adjoint to the canonical map T_1 on $A \otimes A$ and that the generalized inverse R_2 of T_2 for the dual is in turn the adjoint of the generalized inverse R_1 of the original canonical map T_1 .

Now take $a, a' \in A$ and $b, b' \in \widehat{A}$. Then

$$\begin{aligned}\langle E(a \otimes a'), b \otimes b' \rangle &= \langle T_1 R_1(a \otimes a'), b \otimes b' \rangle \\ &= \langle a \otimes a', R_2 T_2(b \otimes b') \rangle \\ &= \langle a \otimes a', (b \otimes 1) F_2(1 \otimes b') \rangle.\end{aligned}$$

We know that $E(1 \otimes a')$ belongs to $\varepsilon_s(A) \otimes A$ and consequently, if we pair with an element b' in the second factor, we get an element of $\varepsilon_s(A)$. From the fact that the pairing is non-degenerate, we obtain that all elements in $\varepsilon_s(A)$ can be obtained as a linear span of such elements. A similar dual result is true for the right hand side.

Then, for all $y \in \varepsilon_s(A)$ there is a y_1 in $\varepsilon_s(\widehat{A})$ so that

$$\langle ya, b \rangle = \langle a, bS(y_1) \rangle$$

for all $a \in A$ and $b \in B$. It is clear that y_1 is uniquely determined by y . Hence we get a linear map $\gamma_s : \varepsilon_s(A) \rightarrow \varepsilon_t(\widehat{A})$ by putting $\gamma_s(y) = S(y_1)$. It is also clear that γ_s is a homomorphism and that it is injective. A similar argument as before will give that it is also surjective. Hence we get an isomorphism $\gamma_s : \varepsilon_s(A) \rightarrow \varepsilon_t(\widehat{A})$ determined by the formula (2.13).

The other result is proven in a completely similar way. We now use that $T_2 R_2(a \otimes a') = (a \otimes a')E$ and that $R_1 T_1(b \otimes b') = (b \otimes 1)F_1(1 \otimes b')$ where $F_1 = (\iota \otimes S)E$. \square

We can obtain partial results like above by using the following argument.

Take $y \in \varepsilon_s(A)$. For $a \in A$ and $b, b' \in \widehat{A}$ we have on the one hand

$$\begin{aligned}\langle ya, bb' \rangle &= \langle \Delta(ya), b \otimes b' \rangle \\ &= \langle (1 \otimes y)\Delta(a), b \otimes b' \rangle \\ &= \langle \Delta(a), b \otimes y \triangleright b' \rangle \\ &= \langle a, b(y \triangleright b') \rangle.\end{aligned}$$

On the other hand, this is equal to $\langle a, y \triangleright (bb') \rangle$. It follows that there is a right multiplier z of \widehat{A} so that $y \triangleright b = bz$ for all $b \in \widehat{A}$. Using similar arguments, one finds that z is actually a multiplier of \widehat{A} . This defines then a homomorphism from $\varepsilon_s(A)$ to $M(\widehat{A})$. It is easy to argue that it is injective and that it maps into the multiplier algebra \widehat{A}_t of the target algebra $\varepsilon_t(\widehat{A})$. However, there seems no direct way along these lines to get that it actually has an image in the target algebra itself.

This result is known for weak Hopf algebras (see e.g. Lemma 2.6 in [2]).

We also get explicit formulas for these isomorphisms γ_s and γ_t in the case of the weak multiplier Hopf algebra associated with a separability idempotent as we study in Section 3. See also Proposition 3.19 in [26].

3. Special cases and examples

In this section, we will give some examples and use these to illustrate the results on integrals obtained in Section 1 and on duality as proven in Section 2. Together with these examples, we also discuss some special cases.

We begin with the two weak multiplier Hopf algebras associated with a groupoid.

Integrals on $K(G)$ for a groupoid G

We consider any groupoid G . For the algebra A we take the complex functions with finite support in G and pointwise product. A coproduct Δ is defined on A by

$$\Delta(f)(p, q) = \begin{cases} f(pq) & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

This is the first weak multiplier Hopf algebra associated with the groupoid. It is Example 1.15 in [28]. Recall that A_s is the algebra of complex functions f on G with the property that $f(p) = f(q)$ whenever p, q are elements in G with the same source. Similarly A_t is the algebra of complex functions f on G where $f(p) = f(q)$ for elements p, q with the same source. See e.g. a remark in the introduction of [29].

For this case, we have the following characterization of left and right integrals. Recall that for a left integral φ , we need $(\iota \otimes \varphi)\Delta(f) \in A_t$ for all f and that for a right integral ψ we need $(\psi \otimes \iota)\Delta(f) \in A_s$ for all f .

3.1. Theorem For any $g \in A_s$, the map $f \mapsto \sum_u g(u)f(u)$ is a left integral and any left integral on A is of that form. Similarly, for any $h \in A_t$, the map $f \mapsto \sum_v h(v)f(v)$ is a right integral and any right integral on A is of that form.

Proof: i) Assume that g is a function on G satisfying $g(p) = g(q)$ if $s(p) = s(q)$. Define φ on A by $\varphi(f) = \sum_u g(u)f(u)$. Remark that this can be defined as we assume that elements $f \in A$ have finite support. Then

$$(\iota \otimes \varphi)\Delta(f)(p) = \sum_u g(u)f(pu) = \sum_u g(pu)f(pu) = \sum_{u'} g(u')f(u')$$

where in the first two sums, we only take the sum over elements u with the property that $t(u) = s(p)$ while in the last sum, we only take elements u' with the property that $t(u') = t(p)$. We see immediately that this is only dependent on the target $t(p)$ and hence $(\iota \otimes \varphi)\Delta(f) \in A_t$.

ii) Conversely, assume that φ is a left integral. As it is a linear functional on A , it is of the form $f \mapsto \sum_u g(u)f(u)$ for a function g on G . Take for f the function that is 1 in an element q and 0 in all other elements. Then

$$(\iota \otimes \varphi)\Delta(f)(p) = \begin{cases} g(p^{-1}q) & \text{if } t(p) = t(q), \\ 0 & \text{otherwise.} \end{cases}$$

If this function of p is only dependent on $t(p)$, that is the source of p^{-1} , we must have that $g(u)$ is only dependent on $s(u)$.

iii) Similarly, for a function h in A_t , the map ψ on A defined by $\psi(f) = \sum_v h(v)f(v)$ will be a right integral and any right integral is of this form. \square

Observe that we can take $g = 1$ and $h = 1$. We will then get a faithful left integral that is also a right integral.

On the other hand, there exists left integrals that are not right invariant and right integrals that are not left invariant. The relation between left integrals and right integrals is found in Proposition 1.6. As in this example the underlying algebra is abelian, there is only one case we have to consider. We illustrate this, and some other properties in the following example. We are still working with $A = K(G)$ as above.

3.2. Example i) Take a left integral φ and a right integral ψ . Then φ has the form $f \mapsto \sum_u g(u)f(u)$ where g is a function in A_s and ψ is of the form $g \mapsto \sum_v h(v)f(v)$ with $h \in A_t$. Now take any function c on G and let $a = gc$ and $b = hc$. Then we have obviously

$$\psi(fa) = \sum_v h(v)(fa)(v) = \sum_v h(v)f(v)a(v) = \sum_v h(v)g(v)c(v)f(v)$$

for all f . A similar argument gives the same result for $\varphi(fb)$ so that $\psi(fa) = \varphi(fb)$ for all $f \in A$. The reader can verify that the expressions for a and b in i) of Proposition 1.6, in terms of the elements p, q , will yield precisely the above choice with c given by $v \mapsto \sum_r p(vr^{-1})q(r)$ (and where the sum is taken over the elements r with the same source as the element v so that vr^{-1} is defined).

ii) Remark that the more natural choice for the function c above would be 1. This will give the strongest relation of the form $\psi(fg) = \varphi(fh)$ for all $f \in A$.

iii) However, in the case e.g. that φ is faithful, that is when g is never 0, we can take $c = g^{-1}$ so that $a = 1$ and we find that $\psi(f) = \varphi(f\delta)$ where $\delta = hg^{-1}$. Now we have $\Delta(\delta) = E(h \otimes g^{-1})$ where E is the canonical idempotent. This illustrates Proposition 1.9.

Similar observations can be made for the relation between two left integrals and two right integrals. If e.g. φ_1 and φ_2 are left integrals, given by functions g_1 and g_2 in A_s respectively, then $\varphi_1(fa) = \varphi_2(fb)$ if $a = g_2c$ and $b = g_1c$ for some c . The strongest choice is again with $c = 1$. If however, say φ_1 is faithful, so that h_1 is invertible, we can take $c = g_1^{-1}$ so that $b = 1$ and we get $\varphi_2(f) = \varphi_1(fa)$ with $a = g_2g_1^{-1}$. In this case, still $a \in A_t$. This illustrates Proposition 1.8.

We can also illustrate the other results. Consider e.g. Proposition 1.3. If φ is a left integral, given by the function g , then for $f \in A$ and $p \in G$ we find

$$((\iota \otimes \varphi)\Delta(f))(p) = \sum_u g(u)f(pu) = \sum_u g(p^{-1}v)f(pp^{-1}v) = \sum_u g(v)f(pp^{-1}v)$$

where the first sum is taken over elements u with $t(u) = s(p)$ and where the following sums taken over the elements v with $t(v) = t(p)$. This last expression is equal to $((\iota \otimes \varphi)\Delta(f))(pp^{-1})$.

Next, let us illustrate the formulas in Proposition 1.5. Again take a function $g \in A_s$ and the left integral φ associated with it. Then for every $f_1, f_2 \in A$ and $p \in G$ we find

$$\begin{aligned} ((\iota \otimes \varphi)((1 \otimes f_1)\Delta(f_2)))(p) &= \sum_u g(u)f_1(u)f_2(pu) \\ &= \sum_v g(p^{-1}v)f_1(p^{-1}v)f_2(v) \\ &= \sum_v g(v)f_1(p^{-1}v)f_2(v) \end{aligned}$$

where the first sum is taken over elements u satisfying $t(u) = s(p)$ and the other sums over element v satisfying $t(v) = t(p)$. The last expression is now $((\iota \otimes \varphi)(\Delta(f_1)(1 \otimes f_2)))(p^{-1})$.

Finally, let us illustrate some of the formulas obtained in Proposition 2.7 of [8] and given again in Proposition 1.4 in Section 1.

Because the algebra is abelian, we have that $S^{-1} = S$ and so $F_1 = F_3$ and $F_2 = F_4$. Hence, we have only one formula for a left integral (cf. Equation (1.3)) and only one formula for a right integral (cf. Equation (1.4)).

If φ is a left integral given by the function $g \in A_s$, we have on the one hand that

$$(\iota \otimes \varphi)\Delta(f)(p) = \sum_u g(u)f(pu) = \sum_u g(v)f(v) \quad (3.1)$$

for all f and all p , where the first sum is taken over elements u with $t(u) = s(p)$ and the second sum over the elements v with the same target as p . On the other hand we get

$$\begin{aligned} (\iota \otimes \varphi)(F_2(1 \otimes f))(p) &= \sum_u ((S \otimes \iota)E)(p, u)f(u)g(u) \\ &= \sum_v f(v)g(v) \end{aligned} \quad (3.2)$$

where the last sum is taken over those elements v that satisfy $(t(v) = t(p))$. We see that the formulas in (3.1) and (3.2) yield the same result. This illustrates Equation (1.3) of Section 1.

In a similar way, we can check the formula for a right integral (Equation (1.3)).

The dual of $K(G)$ for a groupoid G

We again consider a groupoid G and its associated weak multiplier Hopf algebra (A, Δ) with $A = K(G)$ as in the previous item. We will now construct the dual \widehat{A} , denoted also by B , and by doing this, we illustrate the general procedure. As expected, we get the groupoid algebra as given already in Example 1.16 of [28].

Of course, it would also be possible to treat this case independently, just as we did for $K(G)$. But since we want to illustrate the construction of the dual in an very explicit way, we treat it as below.

First we get the dual algebra B in the next proposition.

3.3. Proposition Take the left integral φ on A defined by $f \mapsto \sum_p f(p)$. For any $h \in K(G)$ we consider the linear functional $f \mapsto \varphi(fh)$. This defines a bijective linear map from $K(G)$ to the dual space B . Denote by λ_p the linear functional on $K(G)$ given by $f \mapsto f(p)$. These elements span B and the product in B is given by

$$\lambda_p \lambda_q = \begin{cases} \lambda_{pq} & \text{if } pq \text{ is defined,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof: i) Take $h \in K(G)$. Define $\gamma(h)$ as the linear functional $f \mapsto \varphi(fh)$. Clearly $\gamma(h)$ belongs to B . The map γ is injective because φ is faithful. On the other hand,

when φ_1 is another left integral, then $\varphi_1(f) = \sum_u g(u)f(u)$ for some $g \in A_t$. If now h_1 is any function in $K(G)$, we get

$$\varphi_1(fh_1) = \sum_u g(u)f(u)h_1(u) = \varphi(fh_1)$$

for all f . Because gh_1 is again in $K(G)$, we see that γ is also surjective. This proves the first statement.

ii) For any p and q we find that

$$\langle f, \lambda_p \lambda_q \rangle = \langle \Delta(f), \lambda_p \otimes \lambda_q \rangle = \Delta(f)(p, q)$$

and by the definition of Δ on A we see that $\langle f, \lambda_p \lambda_q \rangle = f(pq)$ when pq is defined and $\langle f, \lambda_p \lambda_q \rangle = 0$ otherwise. This proves that $\lambda_p \lambda_q = \lambda_{pq}$ when pq is defined and $\lambda_p \lambda_q = 0$ if not. \square

We see that the map γ is an isomorphism of the groupoid algebra $\mathbb{C}G$ to B .

In what follows we will identify a function g in $K(G)$ with its image $\gamma(g)$ in B and so we obtain the pairing

$$(f, g) \mapsto \sum_p f(p)g(q)$$

where f, g are functions in $K(G)$ but with f as an element in A and g as an element in B .

It is not hard to verify that the groupoid algebra is idempotent, and in fact, that it has local units. Indeed, given any element $b := \sum_p g(p)\lambda_p$ in \mathbb{C} , we can consider the sums c and d of the elements λ_e and λ_f where respectively, the first sum is taken of the sources of the elements in the support of g and the second one with the targets of these elements. Then $bc = c$ and $db = b$.

In particular, the algebra is non-degenerate. It has a unit if and only if the set of units in the groupoid is finite.

In the next proposition, we obtain the coproduct on the dual B using the general procedure described in the previous section.

3.4. Proposition The coproduct on B , dual to the product on A , as obtained in general in Proposition 2.7 satisfies and is characterized by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all $p \in G$.

Proof: Let us look at the map T_1 on $B \otimes B$. For all $f \in K(G \otimes G)$ we find

$$\langle f, T_1(\lambda_p \otimes \lambda_q) \rangle = T_2(f)(p, q) = f(p, pq)$$

if pq is defined. Otherwise, we get 0. Hence $T_1(\lambda_p \otimes \lambda_q) = \lambda_p \otimes \lambda_{pq}$ if pq is defined and we get 0 otherwise. It follows that $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all p . \square

It is immediately clear from this property that Δ is a homomorphism and that it is coassociative. Also fullness of Δ is a consequence. In this case, the coproduct is coabelian. For the antipode, we obviously have $S(\lambda_p) = \lambda_{p^{-1}}$. The square of the antipode is again the identity map. For the source and target maps and the source and target algebras, we have the following result.

3.5. Proposition The source and target maps on the dual B are given by

$$s(\lambda_p) = \lambda_{s(p)} \quad t(\lambda_p) = \lambda_{t(p)}$$

for all p . The source and target algebras $\varepsilon_s(B)$ and $\varepsilon_t(B)$ are the same and spanned by elements λ_e where e is a unit. Their multiplier algebras B_s and B_t are identified with all complex functions on the units, with pointwise product.

We can now illustrate Proposition 2.19.

For all $y \in \varepsilon_s(A)$ we should have an element $\gamma_s(y)$ in the target algebra of B satisfying $\langle ya, b \rangle = \langle a, b\gamma_s(y) \rangle$ for all $a \in A$ and $b \in B$. Take any element $a \in A$ while $b = \lambda_p$ where $p \in G$. For the left hand side we get $a(p)y(p)$ and for the right hand side $\sum_r a(r)(b\gamma_s(y))(r)$. This must hold for all a and it follows that $(b\gamma_s(y))(p) = y(p)$ and that it is 0 in other points. Because $b = \lambda_p$, this means that $\gamma_s(y)$ must have support in the units and that in fact $\gamma_s(y)(e) = y(r)$ if $e = s(r)$. This is well-defined because it is assumed that $y(r) = y(r')$ if r and r' have the same source. One can verify that the map γ_s is indeed an isomorphism from the source algebra of A to the target algebra of B .

Similarly the isomorphism γ_t of the target algebra of A to the source algebra of B is given here by $\gamma_t(x)(e) = x(r)$ when $e = t(r)$.

The general formula (cf. 2.10) for the dual counit yields that $\varepsilon(\lambda_p) = 1$ for all p in G . This is indeed the unique counit on the dual (B, Δ) . In the following proposition, we obtain the dual canonical idempotent \hat{E} .

3.6. Proposition The canonical idempotent for the dual is $\sum_e \lambda_e \otimes \lambda_e$ where the sum is taken over the units of G .

Proof: We have seen in 2.13 that \hat{E} is given by the formula $\langle a \otimes a', \hat{E} \rangle = \varepsilon(aa')$ where ε is the counit of A . The counit on $K(G)$ is obtained by taking the sum of the values in the units. So here we get the sum $\sum_e a(e)a'(e)$ over the units e . This means that, as a function of two variables, we have $\hat{E}(e, e) = 1$ and 0 in all other points. Hence $\hat{E} = \sum_e \lambda_e \otimes \lambda_e$. \square

Of course, this is what we expect as the identity 1 in $M(B)$ is the sum $\sum_e \lambda_e$ and so $\Delta(1) = \sum_e \lambda_e \otimes \lambda_e$.

It is instructive to verify that this sum indeed defines a multiplier of $B \otimes B$ and that it is the smallest idempotent in $M(B \otimes B)$ with the property that $\hat{E}\Delta(b) = \Delta(b)$ and $\Delta(b)\hat{E} = \Delta(b)$. For this, it is enough to look at elements b of the form λ_p with $p \in G$. Similarly, one can verify that $\hat{E}(B \otimes B) = \Delta(B)(1 \otimes B) = \Delta(B)(B \otimes 1)$ as well as $(B \otimes B)\hat{E} = (1 \otimes B)\Delta(B) = (B \otimes 1)\Delta(B)$.

Integrals on the dual

It is not hard to find the integrals on the dual B directly from the definition of the coproduct Δ on B as obtained in Proposition 3.4. However, following the spirit of this section, we will obtain the integrals from the general result we found in the section on duality, in Proposition 2.16. Remark that left integrals are also right integrals and vice versa because the coproduct is coabelian.

3.7. Proposition If g is a function on G with support in the set of units, then $\lambda_p \mapsto g(p)$ is an integral on B . Any integral is of this form.

Proof: i) As in Proposition 2.16 take $a \in A$ and define ψ_a on B by

$$\psi_a(\omega) = \varphi(a\varepsilon_s(c)) \quad \text{when} \quad \omega = \varphi(\cdot c)$$

where φ is a left integral on A and c an element A . We can take for φ the integral $f \mapsto \sum_p f(p)$. Given the identification of functions in $K(G)$ with elements in B , we get $\psi_a(f) = \sum_u a(u)f(s(u))$, in particular $\psi_a(\lambda_p) = \sum_u a(u)$ where the sum is taken over the elements u with $s(u) = p$. We see that $\psi_a(\lambda_p)$ is 0 except if p is a unit.

ii) Conversely, for any function g on G with support in the set of units, the map $\psi : \lambda_p \mapsto g(p)$ will be an integral. Indeed, for such a function we have

$$(\psi \otimes \iota)\Delta(\lambda_p) = g(p)\lambda_p$$

and this belongs to B_s as this is precisely the algebra of functions with support in the units. \square

Also here, we have a faithful integral:

3.8. Proposition Define φ on B by $\varphi(\lambda_e) = 1$ when e is a unit and $\varphi(\lambda_p) = 0$ otherwise. Then φ is a faithful integral.

Proof: We know already that φ , defined as in the formulation of the proposition, is an integral. We just have to show that this one is faithful.

So, let b be any element in B and assume that $\varphi(\lambda_p b) = 0$ for all p . Now, given p , we have that pq is a unit if and only if $q = p^{-1}$. It follows that $b(p^{-1}) = 0$. This holds for all p and so $b = 0$. Similarly on the other side. \square

The weak multiplier Hopf algebra associated with a separability idempotent

In this item we briefly treat a special example of a weak multiplier Hopf algebra associated with a separability idempotent. In particular, the dual weak multiplier Hopf algebra for this example is constructed. However, we do not give details here. We refer to a separate note where this example and its dual are treated with full details (see [26]).

The starting point is a pair B, C of non-degenerate and idempotent algebras and a separability idempotent $E \in M(B \otimes C)$. Recall Definition 1.4 from [24].

3.9. Definition Let E be an idempotent in the multiplier algebra $M(B \otimes C)$ and assume that $E(1 \otimes b)$ and $(c \otimes 1)E$ belong to $B \otimes C$ for all $b \in B$ and $c \in C$. We assume that E is full in the sense that the left leg and the right leg of E are respectively all of B and all of C . Furthermore it is required that there are non-degenerate anti-homomorphisms $S_B : B \rightarrow M(C)$ and $S_C : C \rightarrow M(B)$ satisfying (and characterized by)

$$E(b \otimes 1) = E(1 \otimes S_B(b)) \quad \text{and} \quad (1 \otimes c)E = (S_C(c) \otimes 1)E$$

for all $b \in B$ and $c \in C$. Then E is called a separability idempotent.

If the maps S_B and S_C have range in C and B respectively, E is called semi-regular and if moreover these maps are anti-isomorphisms, then E is called regular. We will mostly assume that E is regular.

The regular case was first studied in the first version of [24], while the more general case is considered in the second version of this paper.

We denote by φ_B and φ_C the distinguished linear functionals on B and C satisfying

$$(\varphi_B \otimes \iota)E = 1 \quad \text{and} \quad (\iota \otimes \varphi_C)E = 1$$

in $M(C)$ and $M(B)$ respectively. Recall that in the regular case, the distinguished functionals are faithful and have KMS automorphisms (denoted by σ_B and σ_C respectively), see Proposition 2.8 in [24].

We have the following associated weak multiplier Hopf algebra (cf. Proposition 3.2 in [29]).

3.10. Proposition Let A be the algebra $C \otimes B$. Define Δ on A by

$$\Delta(c \otimes b) = c \otimes E \otimes b$$

for all $b \in B$ and $c \in C$. Then we have a weak multiplier Hopf algebra.

The counit is given by $\varepsilon(c \otimes b) = \varphi_B(S_C(c)b) = \varphi_C(cS_B(b))$. The antipode satisfies $S(c \otimes b) = S_B(b) \otimes S_C(c)$ and the source and target maps are

$$\varepsilon_s(c \otimes b) = 1 \otimes S_C(c)b \quad \text{and} \quad \varepsilon_t(c \otimes b) = cS_B(b) \otimes 1$$

for all b, c .

We see that $\varepsilon_s(A) = 1 \otimes B$ and $\varepsilon_t(A) = C \otimes 1$. The multiplier algebras A_s and A_t are $1 \otimes M(B)$ and $M(C) \otimes 1$ respectively.

Observe also that A is regular if and only if E is regular (in the sense of Definition 2.3 of [24]). See Proposition 3.3 in [29].

In what follows we will look at A as the algebra generated by B and C with the condition that elements of B commute with elements of C . We can do this by looking at the natural non-degenerate homomorphisms from B and C into $M(A)$ and by identifying B and C with their images. Then A is spanned by elements of the form cb and $cb = bc$ in A . This practice is standard.

Then we write e.g. $\Delta(cb) = (c \otimes 1)E(1 \otimes b)$ and the canonical idempotent of A is E , considered as sitting in $M(A \otimes A)$. We refer to Section 3 in [29].

We get the following characterization of the integrals. See Proposition 2.2 in [26].

3.11. Proposition For any linear functional g on B we have a left integral φ on A given by $\varphi(cb) = \varphi_C(c)g(b)$ and any left integral has this form. Similarly, for any linear functional f on C we have a right integral ψ on A given by $\psi(cb) = f(c)\varphi_B(b)$ and again any right integral is of this form. Moreover, the map $cb \mapsto \varphi_C(c)\varphi_B(b)$ is a faithful left integral that is also a right integral.

The proof of this result is straightforward, again see [26].

The dual of (CB, Δ)

Because we have a faithful integral φ on A , defined by $\varphi(cb) = \varphi_C(c)\varphi_B(b)$, we have that \widehat{A} , as a vector space, can be identified with $C \otimes B$ via the map $v \otimes u \mapsto \varphi(\cdot vu)$. Remark that $\varphi(cbvu) = \varphi_C(cv)\varphi_B(bu)$ for all $b, u \in B$ and $c, v \in C$. So this map is $v \otimes u \mapsto \varphi_C(\cdot v) \otimes \varphi_B(\cdot u)$.

However, in what follows, we will make another identification. It turns out to give nicer formulas.

3.12. Definition We define a pairing of vector spaces A and $B \otimes C$ by

$$\langle cb, u \otimes v \rangle = \varphi_B(bS_C(v))\varphi_C(S_B(u)c).$$

Observe the difference with the formula for the counit on A . Indeed, for the counit we have $\varepsilon(cb) = \varphi_B(S_C(c)b) = \varphi_C(cS_B(b))$.

Because we have

$$\begin{aligned} \varphi_B(bS_C(v))\varphi_C(S_B(u)c) &= \varphi_B(bS_C(v))\varphi_C(c\sigma_C(S_B(u))) \\ &= \varphi(cbS_C(v)\sigma_C(S_B(u))), \end{aligned}$$

we see that

$$u \otimes v \mapsto \varphi(\cdot S_C(v)\sigma_C(S_B(u)))$$

gives a linear isomorphism from the space $B \otimes C$ to \widehat{A} .

In what follows, we will make this identification, but we will use $u \diamond v$ for $u \otimes v$ when we consider it as an element in \widehat{A} . We will also systematically use the letters u, u', \dots and $v, v' \dots$ for elements in B and C respectively, when $B \otimes C$ is identified with \widehat{A} via the pairing in Definition 3.12 above.

For the product on the dual \widehat{A} we find the following result. See Proposition 3.2 in [26].

3.13. Proposition Let $u, u' \in B$ and $v, v' \in C$. For the product in \widehat{A} we find

$$(v \diamond u)(v' \diamond u') = \varepsilon(v'u) v \diamond u',$$

where ε is the counit on A

Again the proof is straightforward and it is found in [26].

By the general theory, we know that this product is associative and non-degenerate. Here, we can verify that the product is associative by a simple calculation. We see also that it is non-degenerate because the map $(u, v) \mapsto \varepsilon(vu)$ is a non-degenerate pairing of the space B with C . This follows from the definition $\varepsilon(vu) = \varphi_B(S_C(v)u)$, the fact that φ_B is faithful and that S_C is bijective.

In what follows, we will also write $B \diamond C$ for the algebra we obtain in Proposition 3.13. The algebra is a (possibly) *infinite matrix algebra*, build with two vector spaces and a non-degenerate pairing. The algebra structure is not dependent on the multiplications in B and C . It only depends on the pairing of the underlying vector spaces.

Because the algebra $B \diamond C$ is non-degenerate, we can consider its multiplier algebra $M(B \diamond C)$. We get the following characterization. See Proposition 3.3 in [26].

3.14. Proposition A linear map $\gamma : B \rightarrow B$ is called *adjointable* if there is a linear mapping $\gamma^t : C \rightarrow C$ satisfying $\varepsilon(v\gamma(u)) = \varepsilon(\gamma^t(v)u)$ for all $u \in B$ and $v \in C$. The maps γ and γ^t determine each other. For any adjointable map $\gamma : B \rightarrow B$ with adjoint γ^t there is a multiplier m of $B \diamond C$ given by

$$m(u \diamond v) = \gamma(u) \diamond v \quad \text{and} \quad (u \diamond v)m = u \diamond \gamma^t(v). \quad (3.3)$$

Any multiplier is of this form.

The embedding of $B \diamond C$ in $M(B \diamond C)$ is found by associating the linear map $u' \mapsto \varepsilon(vu')u$ to the element $u \diamond v$. Its adjoint is $v' \mapsto \varepsilon(v'u)v$. The identity map from B to itself is of course adjointable and the associated multiplier is the identity 1 in $M(B \diamond C)$.

The coproduct on $B \diamond C$

By definition, the coproduct on \widehat{A} is dual to the product on A . Because A is the tensor product of the algebras C and B , it is expected that the coproduct on the dual is also a tensor product of coproducts on the factors. In this case, it would mean that the coproduct on $B \diamond C$ has the form

$$\Delta(u \diamond v) = \sum_{(u),(v)} (u_{(1)} \diamond v_{(1)}) \otimes (u_{(2)} \diamond v_{(2)})$$

where we have the Sweedler notation for coproducts Δ_B on B and Δ_C on C .

This is indeed the case and it is made precise with the following results.

3.15. Proposition Define a pairing of C with B by the formula $\langle c, u \rangle_1 = \varphi_C(S_B(u)c)$. For this pairing we have

$$\langle cc', u \rangle_1 = \langle c \otimes c', \Delta_B(u) \rangle_1$$

for $c, c' \in C$ and $u \in B$, where $\Delta_B(u) = F_1(1 \otimes u)$ and $F_1 = (\iota \otimes S_C)E$ as in Section 1.

3.16. Proposition Define a pairing of B with C by the formula $\langle b, v \rangle_2 = \varphi_B(bS_C(v))$. For this pairing we have

$$\langle bb', v \rangle_2 = \langle b \otimes b', \Delta_C(v) \rangle_2$$

where $\Delta_C(v) = (v \otimes 1)F_2$ for $b, b' \in B$ and $v \in C$ and $F_2 = (S_B \otimes \iota)E$ as in Section 1.

Now we get from these two properties the formula for the coproduct on $B \diamond C$.

3.17. Proposition The coproduct on $B \diamond C$, defined by the formula

$$\langle aa', u \diamond v \rangle = \langle a \otimes a', \Delta(u \diamond v) \rangle$$

for $a, a' \in A$ and $u \in B$ and $v \in C$, is

$$\Delta(u \diamond v) = \sum_{(u),(v)} (u_{(1)} \diamond v_{(1)}) \otimes (u_{(2)} \diamond v_{(2)})$$

where we use the Sweedler notations

$$\sum_{(u)} u_{(1)} \otimes u_{(2)} = \Delta_B(u) = F_1(1 \otimes u)$$

$$\sum_{(v)} v_{(1)} \otimes v_{(2)} = \Delta_C(v) = (v \otimes 1)F_2.$$

We need to make a couple of remarks about this formula.

3.18. Remark i) It is rather remarkable that the pairings $(c, u) \mapsto \langle c, u \rangle_1$ and $(b, v) \mapsto \langle b, v \rangle_2$, that we defined in the Propositions 3.15 and 3.16, yield coproducts $\Delta_B : B \rightarrow B \otimes B$ and $\Delta_C : C \rightarrow C \otimes C$.

ii) Moreover, these coproducts are known. See e.g. Proposition 2.9 in [24].

iii) It is also obvious that the coproducts are homomorphisms. As a consequence, the tensor coproduct, as defined in Proposition 3.17 will be a homomorphism on the tensor product algebra $B \otimes C$. However, this is different from the product in $B \diamond C$ and, from the general theory, we know that this tensor coproduct is also a homomorphism on $B \diamond C$.

The regular weak multiplier Hopf algebra $B \diamond C$

One can now systematically verify that indeed, the algebra $B \diamond C$ as obtained in Proposition 3.13, together with the coproduct given in Proposition 4.16, satisfies all the requirements of a regular weak multiplier Hopf algebra. This is done in detail in Section 3 of [26].

For the counit $\widehat{\varepsilon}$ on the dual we find that $\widehat{\varepsilon}(u \diamond v) = \varphi_B(u)\varphi_C(v)$. This is no surprise because φ_B and φ_C are the counits for the coalgebras (B, Δ_B) and (C, Δ_C) .

For the antipode \widehat{S} on the dual we also get the expected formula. We have

$$\widehat{S}(u \diamond v) = S_B^{-1}(v) \diamond S_C^{-1}(u)$$

for $u \in B$ and $v \in C$. This result is proven by taking the adjoint of the antipode in the pairing. See Proposition 3.10 in [26].

One can verify that \widehat{S} is an anti-isomorphism of the algebra \widehat{A} and that it flips the coproduct. See Propositions 3.11 and 3.12 in [26].

The canonical idempotent is more difficult to handle because it is an element in the multiplier algebra $M(B \diamond C)$ of $B \diamond C$. There is the following result (see Proposition 3.13 in [26]).

3.19. Proposition The maps $\gamma : u \otimes u' \mapsto (uu' \otimes 1)F_1$ from $B \otimes B$ to itself and $\gamma' : v \otimes v' \mapsto F_2(1 \otimes vv')$ from $C \otimes C$ to itself are adjoint to each other (with respect to the tensor product pairing). They are projection maps. And they give the canonical idempotent \widehat{E} in $M(\widehat{A} \otimes \widehat{A})$.

That the maps are idempotent follows simply from the fact that $m_B F_1 = E_{(1)} S_C(E_{(2)}) = 1$ as well as $m_C F_2 = S_B(E_{(1)}) E_{(2)}$. Here we use m_B and m_C for the multiplication maps on $B \otimes B$ and $C \otimes C$ respectively.

Using the formula for the antipode, we find the following formulas for the source and target maps on the dual.

3.20. Proposition For all $u \in B$ and $v \in C$ we have

$$\varepsilon_s(u \diamond v) = \varphi_B(u)(E_{(1)} \diamond E_{(2)}v) \quad \text{and} \quad \varepsilon_t(u \diamond v) = \varphi_C(v)(uE_{(1)} \diamond E_{(2)})$$

where we again use the Sweedler type notation for E .

Recall that $(u \otimes 1)E \in B \otimes C$ when $u \in B$ and this gives the element $uE_{(1)} \diamond E_{(2)}$ in $B \diamond C$. Similarly $E(1 \otimes v) \in B \otimes C$ when $v \in C$ and this gives the element $(E_{(1)} \diamond E_{(2)})v$ in $B \diamond C$.

One can verify that the maps $v \mapsto E_{(1)} \diamond E_{(2)}v$ and $u \mapsto uE_{(1)} \diamond E_{(2)}$ are isomorphisms from the algebras C and B to the source algebras $\varepsilon_s(\hat{A})$ and $\varepsilon_t(\hat{A})$ respectively, see Proposition 3.20 in [26].

They are in fact modified forms of the isomorphisms obtained in general in Proposition 2.19. Again see [26] for the relation between the two results.

Finally we consider the integrals on the dual \hat{A} . From the general theory we find that there is a right integral $\hat{\psi}_c$ on the dual given by $\hat{\psi}_c(u \diamond v) = \varphi_C(S_B(u)cv)$ for all $u \in B$ and $v \in C$ (see Proposition 3.23 in [26]). In fact, we get the following.

3.21. Proposition For any $x \in M(C)$ there is a right integral $\hat{\psi}_x$ on \hat{A} given by $\hat{\psi}_x(u \diamond v) = \varphi_C(S_B(u)xv)$ for all $u \in B$ and $v \in C$. Any right integral is of this form. Similarly, for any $y \in M(B)$ there is a left integral $\hat{\varphi}_y$ on \hat{A} given by $\hat{\varphi}_y(u \diamond v) = \varphi_B(uyS_C(v))$ for all $u \in B$ and $v \in C$. Again any left integral is of this form.

For the proof we refer to Propositions 3.23 and 3.24 in [26]. We also found an expression for the modular element (see Proposition 3.28 in [26]).

3.22. Proposition If $\hat{\varphi}$ is the left integral given by $\hat{\varphi}(u \diamond v) = \varphi_B(uS_C(v))$, then $\hat{\varphi}(S(u \diamond v)) = \hat{\varphi}((u \diamond v)\delta)$ where δ is the multiplier of \hat{A} given by

$$\delta(u \diamond v) = \sigma_B^{-2}(u) \diamond v \quad \text{and} \quad (u \diamond v)\delta = u \diamond \sigma_C^{-2}(v)$$

We finish with a remark. From the formula for the antipode we see that

$$S^2(u \diamond v) = S_B^{-1}S_C^{-1}(u) \diamond S_C^{-1}S_B^{-1}(v) = \sigma_B(u)\sigma_C^{-1}(v)$$

for all u, v . Then using the formulas for δ obtained in the previous proposition, we find

$$S^4(u \diamond v) = \delta^{-1}(u \diamond v)\delta$$

for all u, v . This is a special form of Radford's formula as there is left integral on A that is also right invariant.

4. Conclusion and further research

In this paper, we have continued our work on weak multiplier Hopf algebras, started in [27], [28] and [29], with the study of integrals and duality.

Some questions remain open.

In the examples we considered in Section 4, there always existed a single faithful left integral. It is known from the theory of weak Hopf algebras, that it is possible however

that there is a faithful set of integrals but not a single faithful integral. This follows e.g. from Proposition 2.5 in [7]. We are indebted to G. Böhm for informing us about this example in the paper.

It is known that, for finite-dimensional weak Hopf algebras, a single faithful integral exists if and only if the underlying algebra is Frobenius (see Theorem 3.16 in [2]). For (possibly non finite-dimensional) weak multiplier Hopf algebras, the situation is more subtle. One can expect that when the underlying algebra is Frobenius and when there are enough integrals, there will also exist a single faithful one. Moreover, given the fact that for multiplier Hopf algebras, integrals are unique, and hence the existence of enough integrals implies the existence of a single faithful one, one might wonder if this should eventually only be dependent on the base algebra?

It is expected that the dual also will have a single faithful integral when this is the case for the original algebraic quantum groupoid. The result is true for finite-dimensional weak Hopf algebras (see Theorem 3.16 in [2]). However, it can not immediately be obtained from our result in Proposition 2.17. A finer argument is needed.

In the case of a single faithful integral, we have objects like the Radon-Nikodym elements, giving any left integral in terms of a single faithful left integral (see Proposition 1.8), the modular elements, expressing any right integral in terms of a single faithful integral (see Proposition 1.9) and finally the modular automorphisms of a single faithful left integral (see Proposition 1.7).

One should also prove results like the ones above in the event that there is no single faithful integral.

We have not considered the involutive case in this paper. It can be verified that the dual of a weak multiplier Hopf $*$ -algebra with integrals is again, in a natural way, a weak multiplier Hopf $*$ -algebra with integrals. It is expected that the dual will carry a faithful set of positive integrals when this is the case for the original $*$ -algebraic quantum groupoid. In that case, it should be possible to lift the whole structure to a Hilbert space setting. This is work in progress, cf. [9].

We also have only considered the regular case. There are reasons to believe that the existence of integrals can only be studied under the reasonable condition of regularity, but still, the problem is open. It is not entirely clear what remains possible in the non-regular case. This has not even been studied for multiplier Hopf algebras.

Finally, as we mentioned already in the introduction, while preparing this work, Timmermann obtained results about duality for regular multiplier Hopf algebroids with integrals in [18] and [19]. And he concluded some results about integrals and duality for regular weak multiplier Hopf algebras. Regular multiplier Hopf algebroids were introduced in [20].

The relation between his work and ours has to be investigated further.

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